Institute for Advanced Study

Migration of propellers in Saturn's rings

Bonus: Symplectic Integrators

Hanno Rein @ ISIMA 2011 KIAA Beijing

Symplectic integrators

Observations

Possible explanations

Numerical Integrators

- We want to integrate the equations of motions of a particle

$$\dot{x} = v$$

$$\dot{v} = a(x, v)$$

- For example, gravitational potential

$$a(x) = -\nabla\Phi(x)$$

- In physics, these can usually be derived from a Hamiltonian

$$H = \frac{1}{2}p^2 + \Phi(x)$$

- Symmetries of the Hamiltonian correspond to conserved quantities

Numerical Integrators

- Discretization

$$\dot{x} = v \qquad \longrightarrow \qquad \Delta x = v\,\Delta t$$

$$\dot{v} = a(x, v) \qquad \qquad \Delta v = a(x, v)\,\Delta t$$

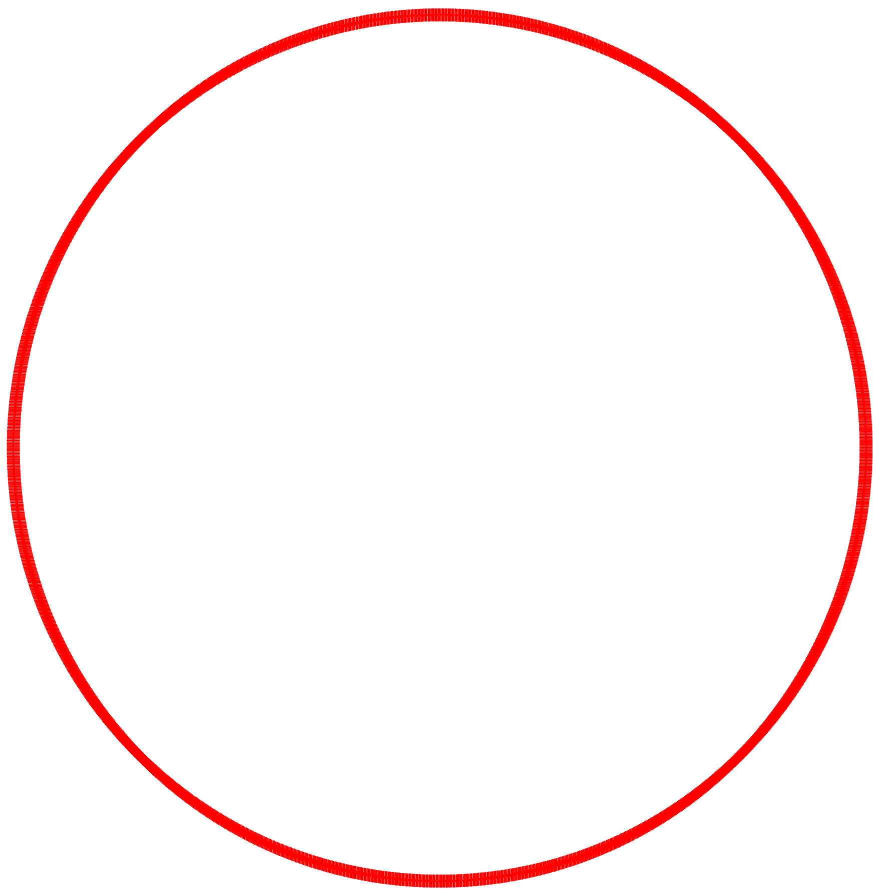
- Hamiltonian

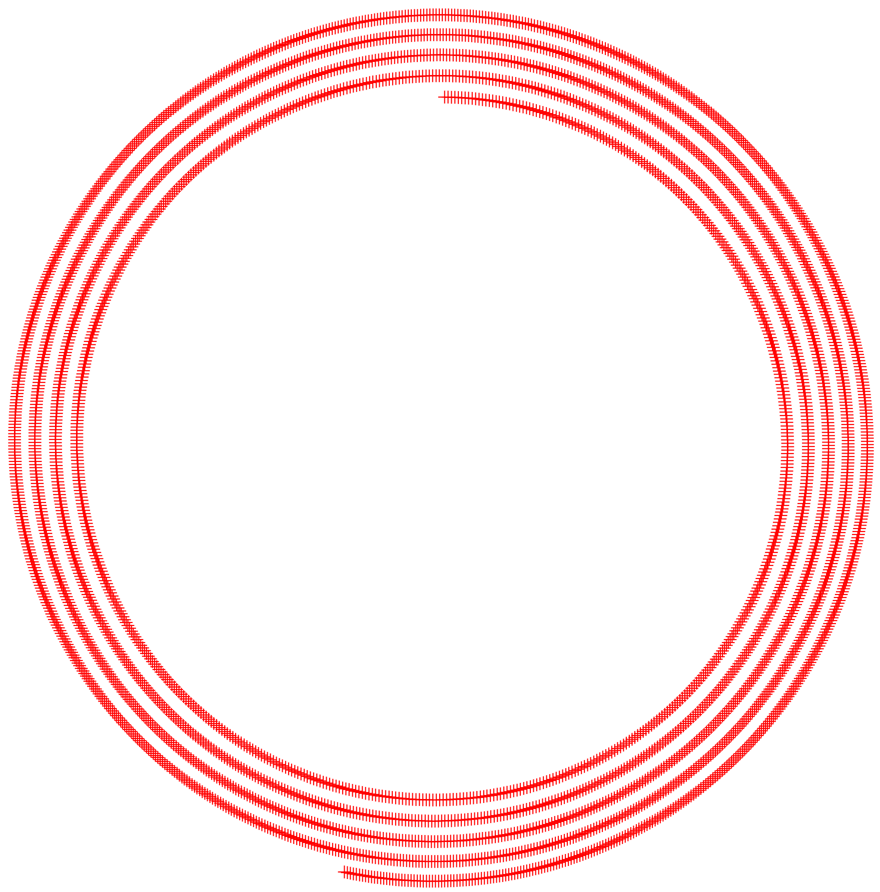
$$H = \frac{1}{2}p^2 + \Phi(x) \qquad \longrightarrow \qquad ?$$

- The system is governed by a 'discretized Hamiltonian', if and only if the integration scheme is symplectic.

- Why does it matter?

Symplectic vs non symplectic integrators





Mixed variable integrators

- So far: symplectic integrators are great.
- How can it be even better?
- We can split the Hamiltonian:

$$H = H_0 + \epsilon\, H_{\text{pert}}$$

Integrate particle exactly with dominant Hamiltonian

Integrate particle exactly under perturbation Hamiltonian

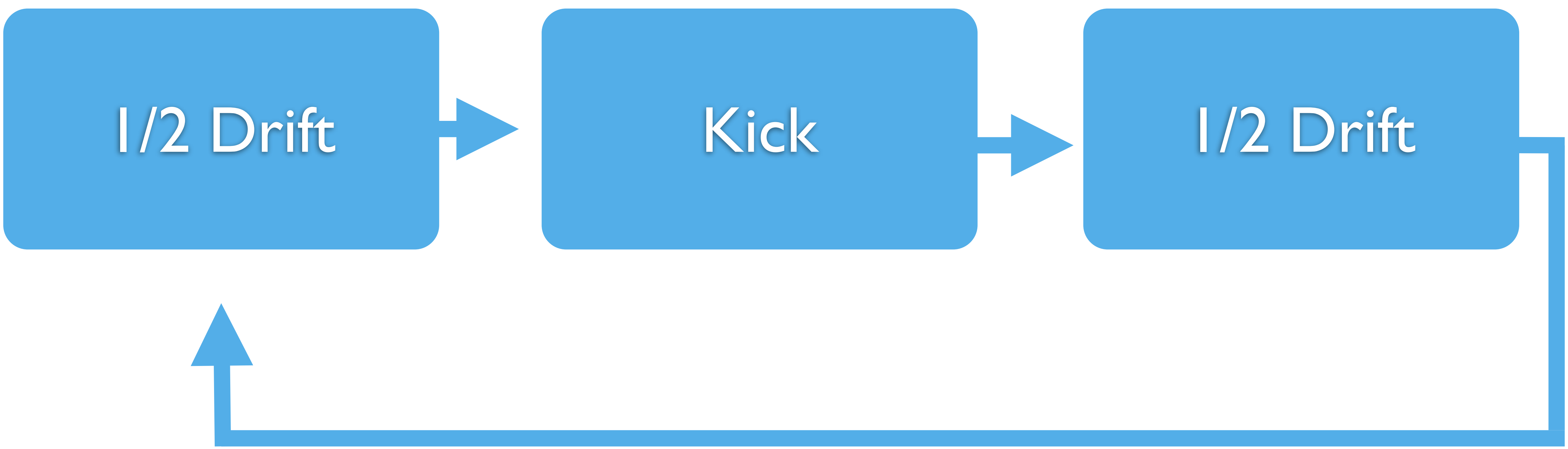
- Switch back and forth between different Hamiltonians
- Often uses different variables for different parts
- Then:

$$\text{Error} = \epsilon\,(\Delta t)^{p+1}\;[H_0, H_{\text{pert}}]$$

Example: Leap-Frog

$$H = \frac{1}{2}p^2 + \Phi(x)$$

Drift Kick



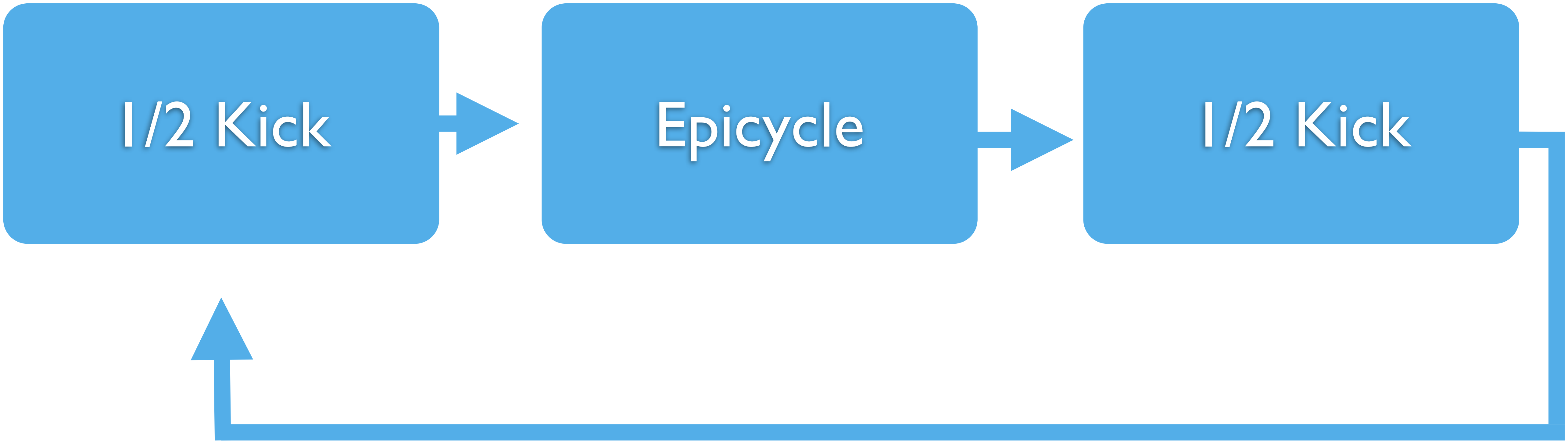
Example: SWIFT/MERCURY

$$H = \frac{1}{2}p^2 + \Phi_{\mathrm{Kepler}}(x) + \Phi_{\mathrm{Other}}(x)$$

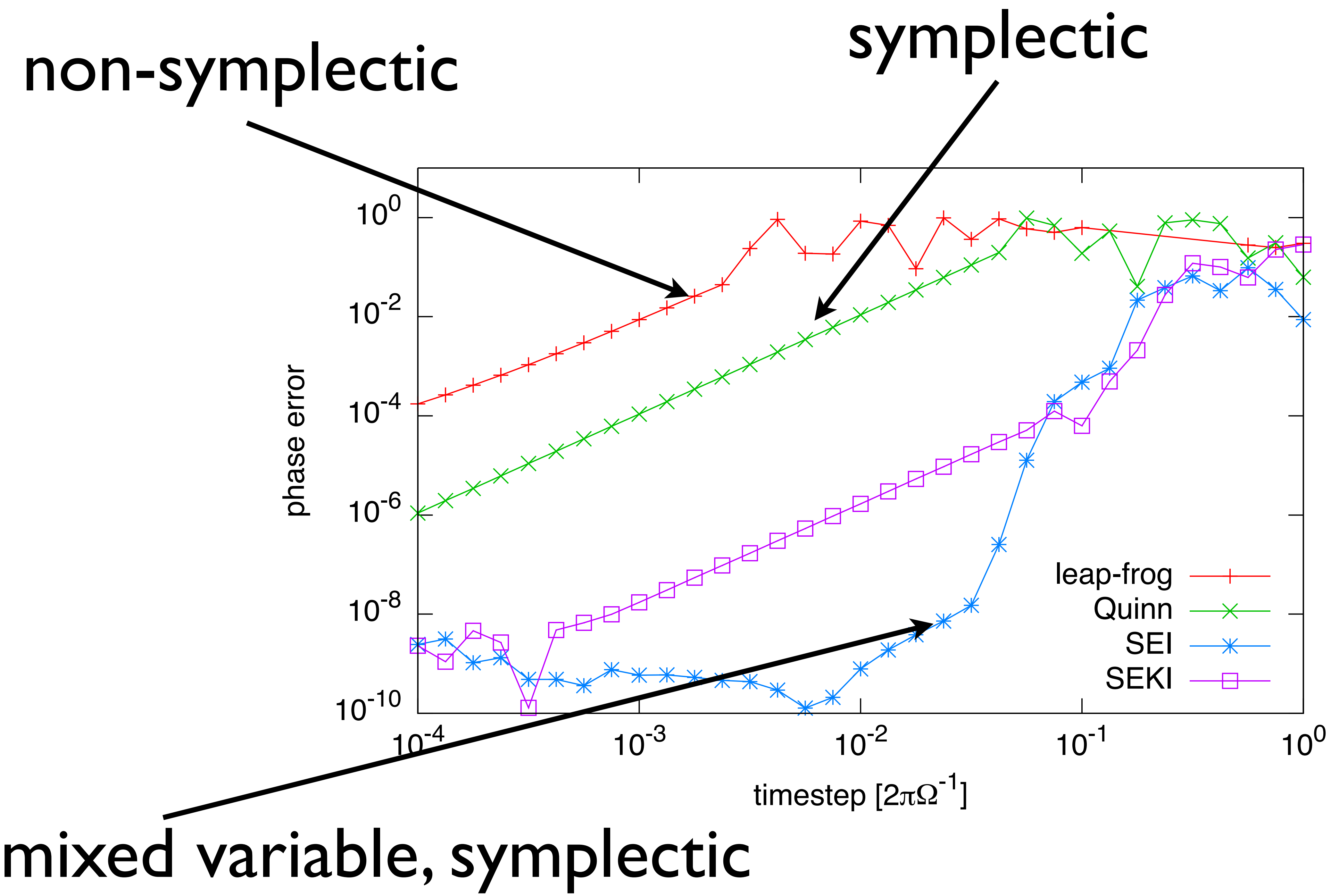
Kepler | Kick

Example: Symplectic Epicycle Integrator

$$H = \underbrace{\frac{1}{2}p^2 + \Omega(p \times r)e_z + \frac{1}{2}\Omega^2\left[r^2 - 3(r \cdot e_x)^2\right]}_{\text{Epicycle}} + \underbrace{\Phi(r)}_{\text{Kick}}$$



10 Orders of magnitude better!



Rein & Tremaine 2011

Conclusions
Part I

Conclusions

Symplectic integrators

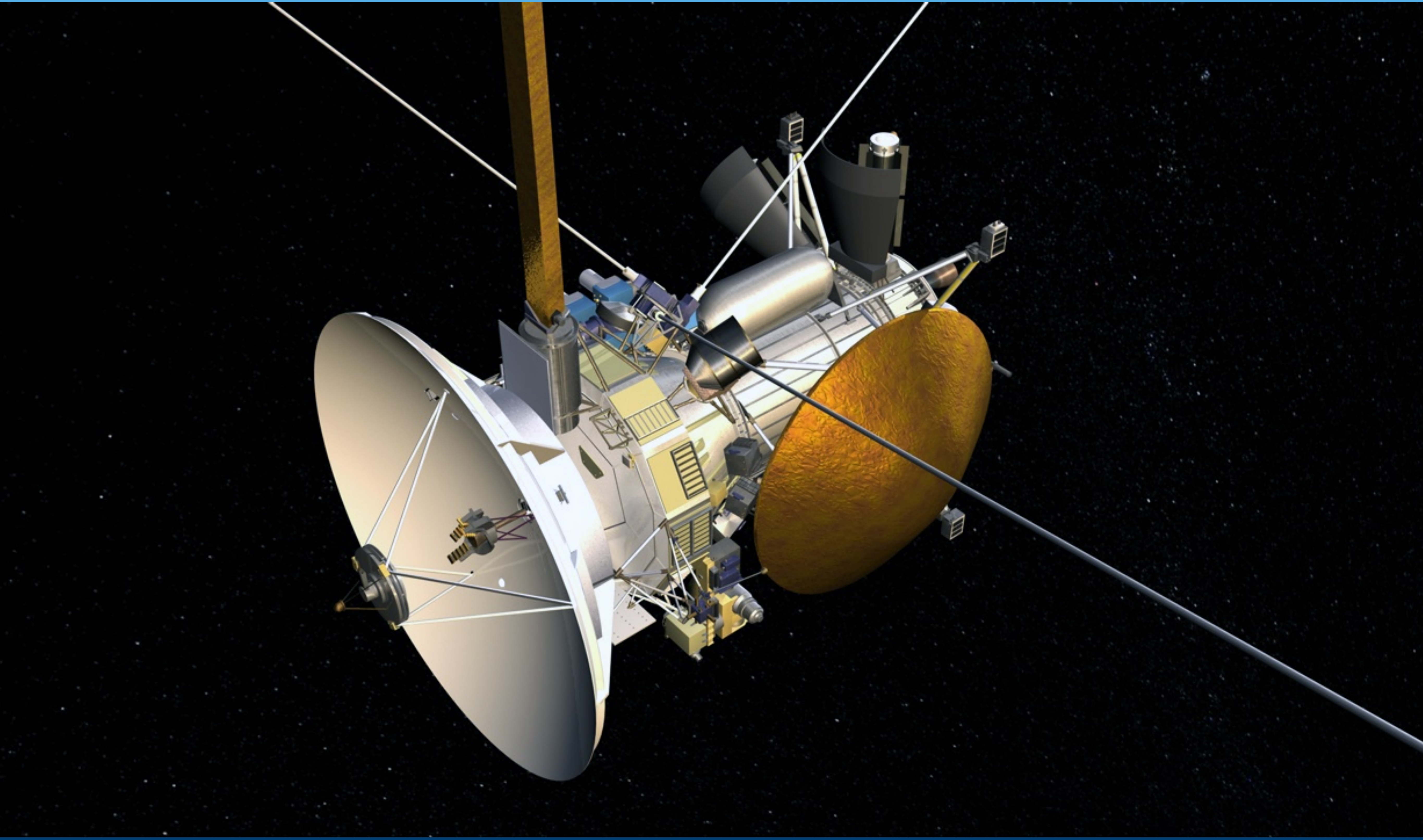
- Hamiltonian systems exhibit large number of symmetries
- These are usually lost in standard integrators (RK)
- Symplectic integrators keep symmetries (might be modify slightly)
- No secular drift
- Excellent performance
- Worth thinking about!

Symplectic integrators

Observations

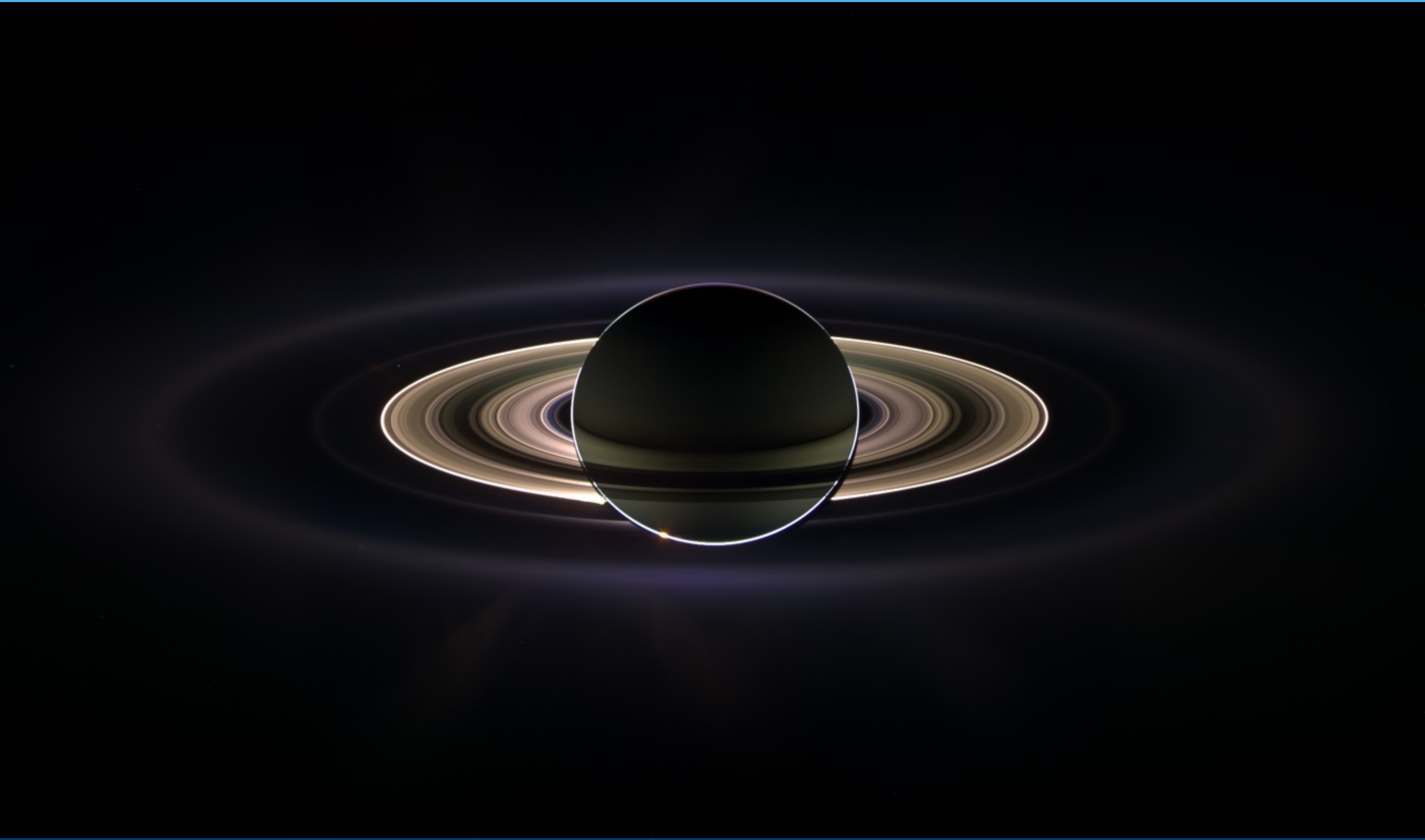
Possible explanations

Cassini spacecraft



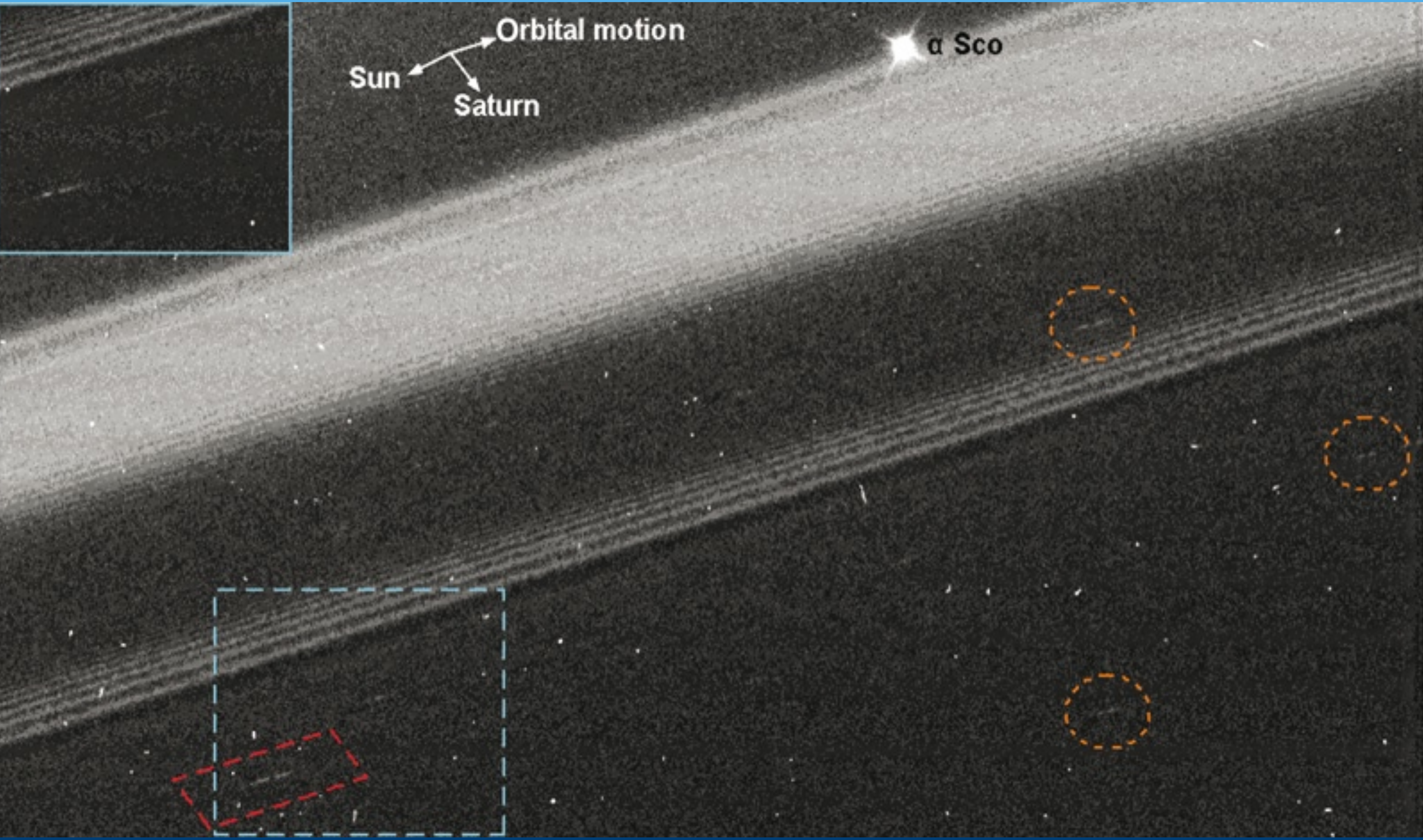
NASA/JPL/Space Science Institute

Cassini spacecraft



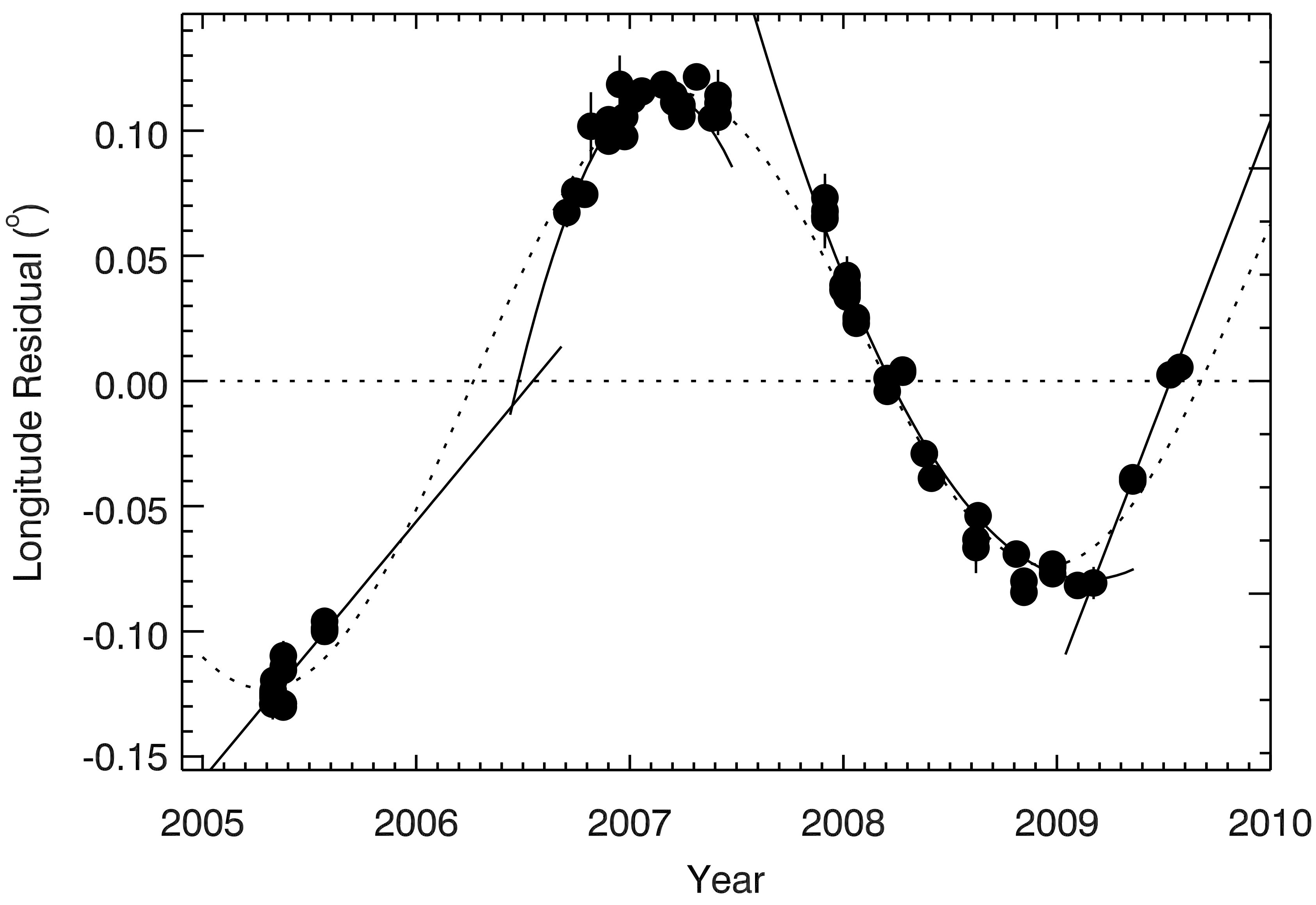
NASA/JPL/Space Science Institute

Propeller structures in A-ring

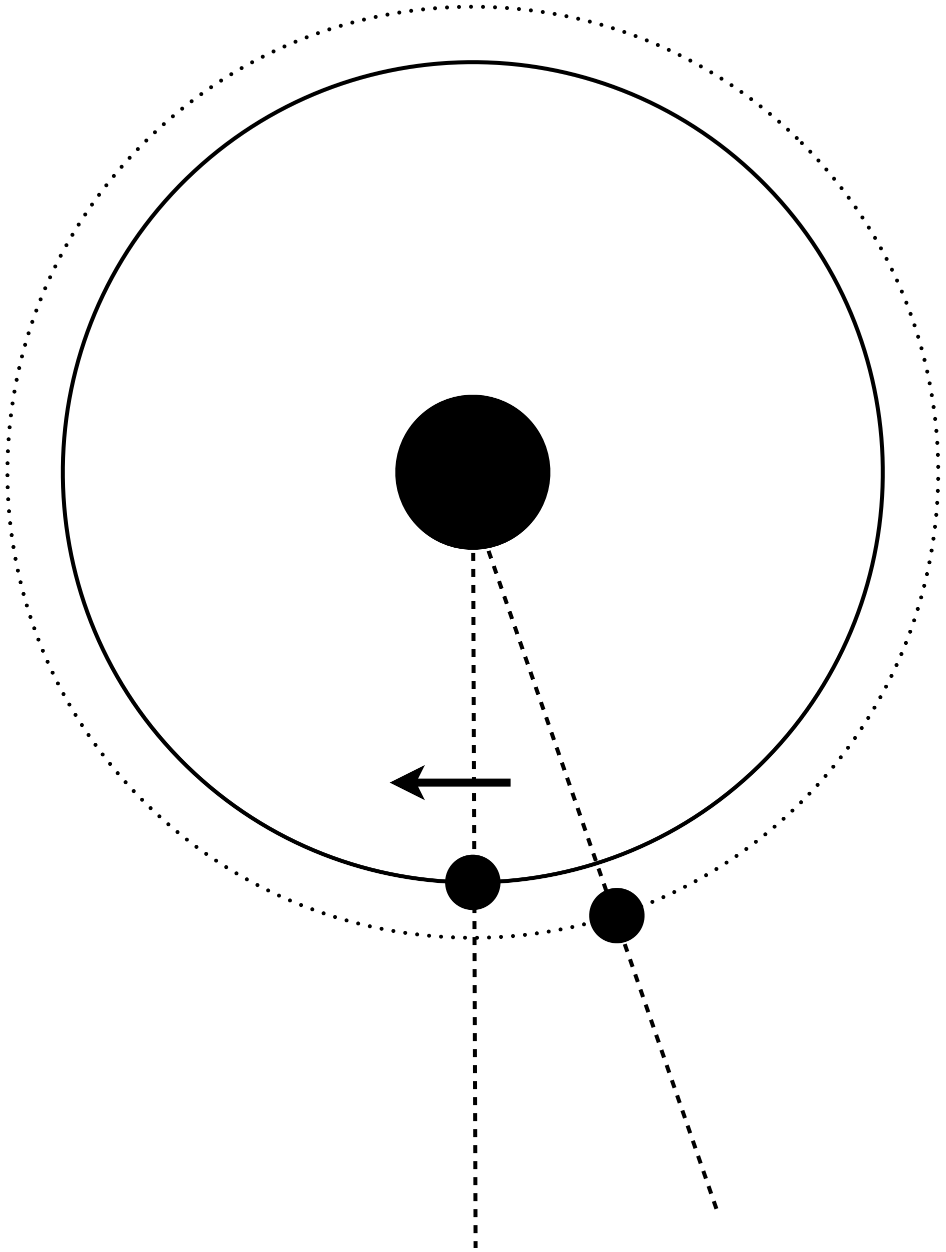


Porco et al. 2007, Sremcevic et al. 2007, Tiscareno et al. 2006

Observational evidence of non-Keplerian motion



Longitude residual



Mean motion [rad/s]

$$n = \sqrt{\frac{GM}{a^3}}$$

Mean longitude [rad]

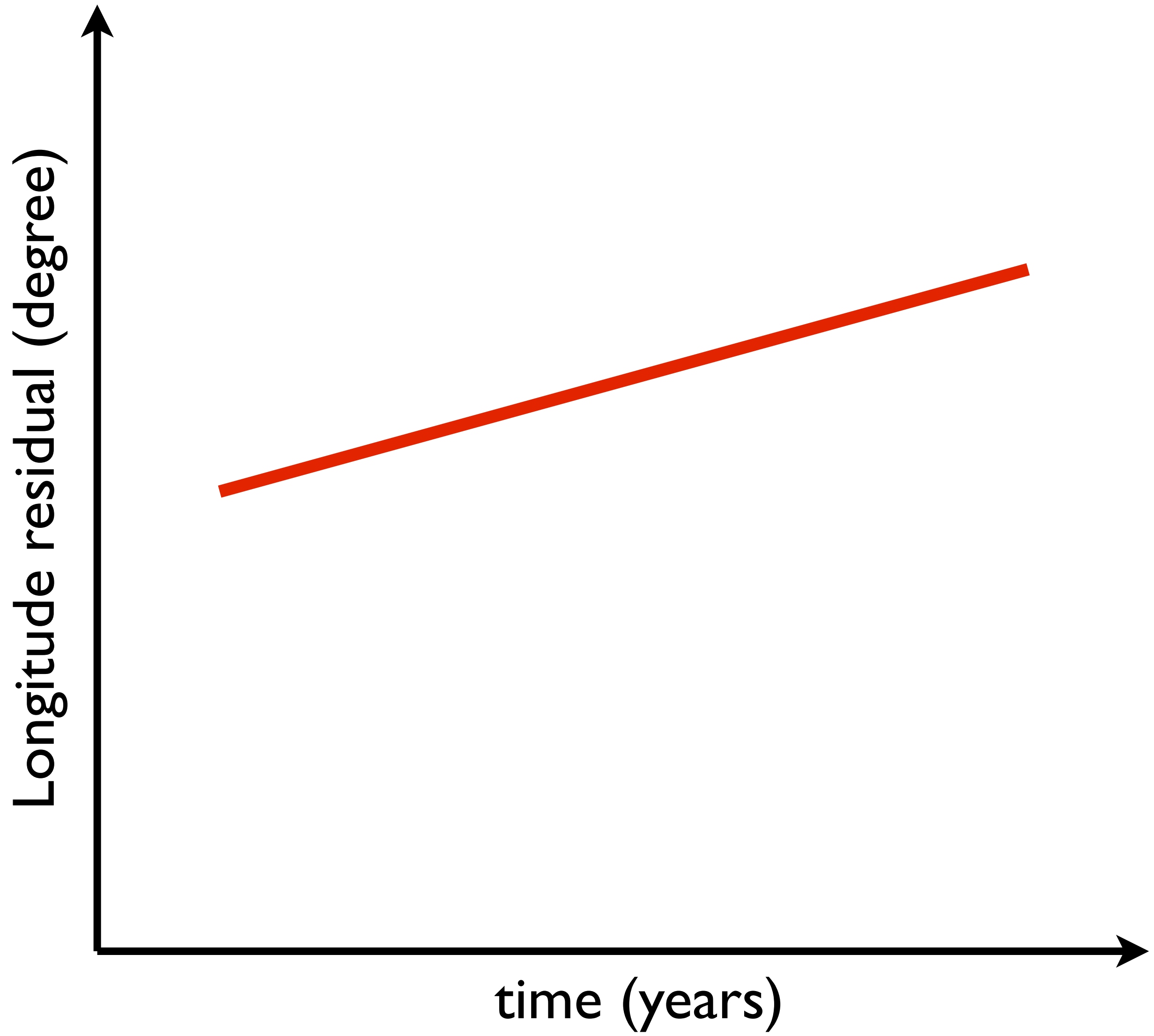
$$\lambda = n\,t$$

$$\lambda(t) - \lambda_0(t) = \int_0^t (n_0 + n'(t'))\,dt' - \underbrace{\int_0^t n_0\,dt'}_{n_0 t}$$

Keplerian rotation: linear

$$n'(t) = const$$

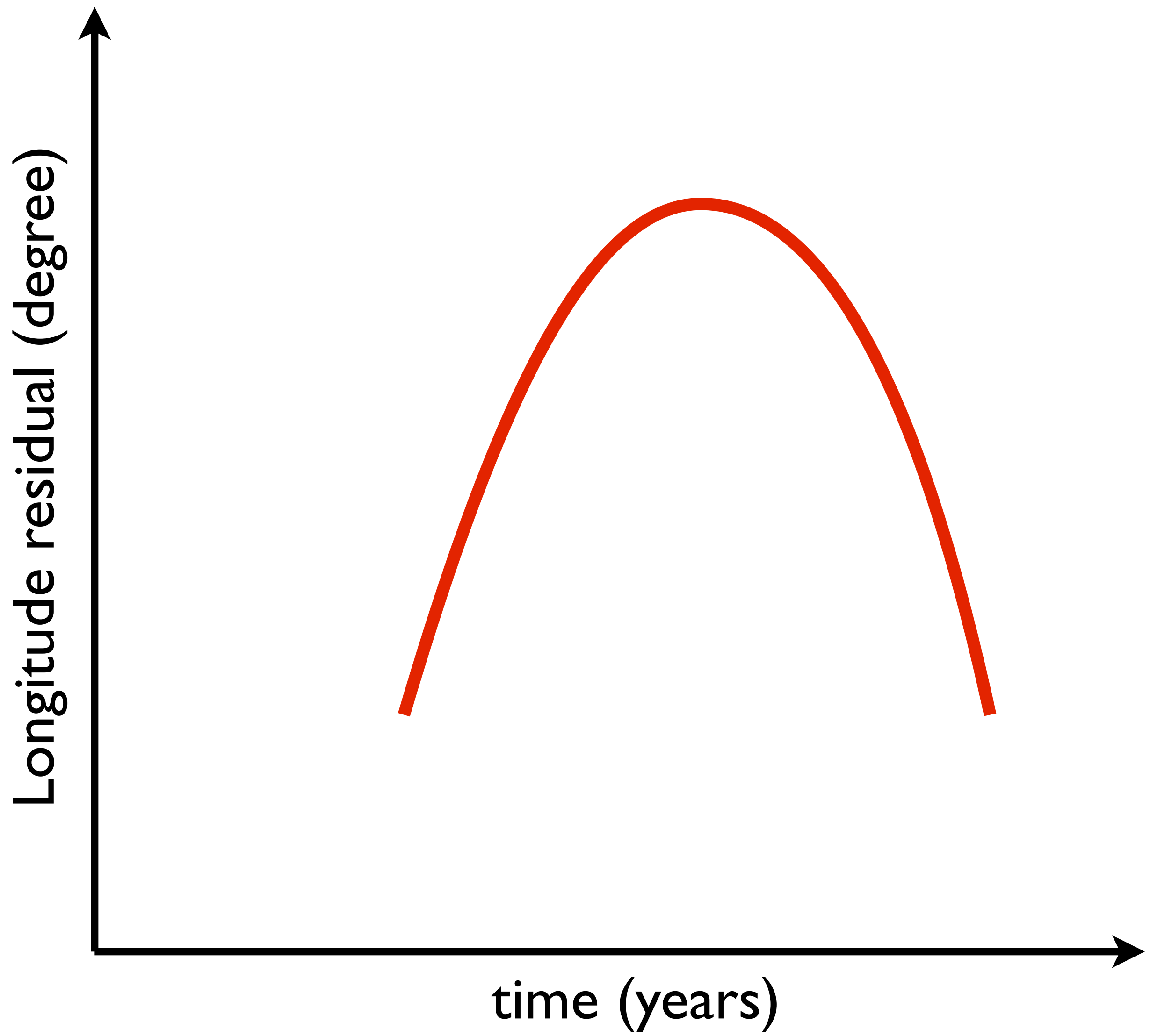
$$\lambda(t) - \lambda_0(t)$$
$$= \int_0^t (n_0 + n'(t'))\, dt'$$
$$- \int_0^t n_0\, dt'$$
$$= n_0\, t + n'\, t - n_0\, t = n'\, t$$



Constant migration rate: quadratic

$$n'(t) = const \cdot t$$

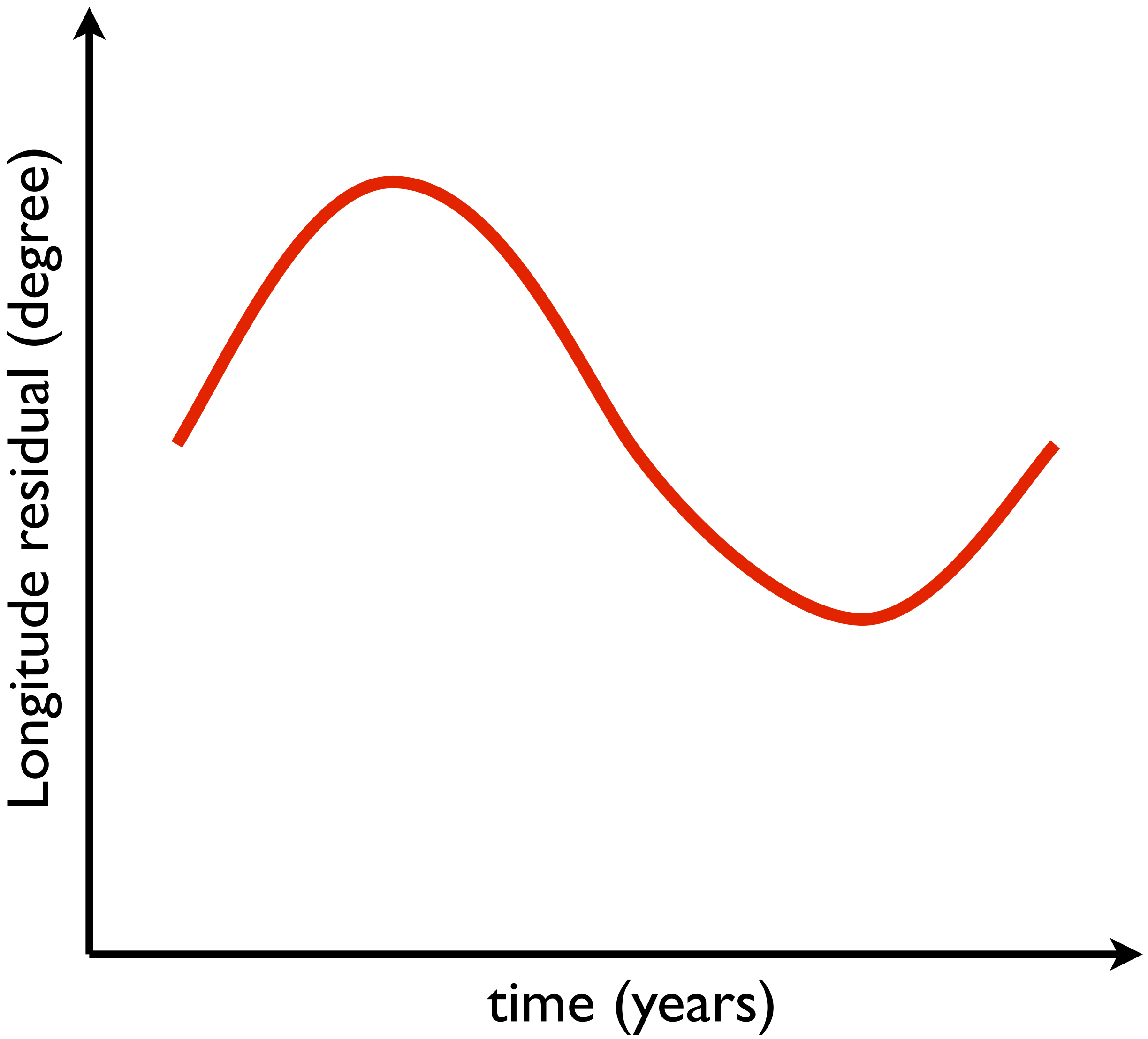
$$\lambda(t) - \lambda_0(t)$$
$$= \int_0^t (n_0 + n'(t'))\, dt'$$
$$- \int_0^t n_0\, dt'$$
$$= \frac{1}{2} const \cdot t^2$$



Resonance: sine-curve

$$n'(t) = cos(t)$$

$$\begin{aligned}
&\lambda(t) - \lambda_0(t) \\
&= \int_0^t (n_0 + n'(t'))\, dt' \\
&\;- \int_0^t n_0\, dt' \\
&= \sin(t)
\end{aligned}$$



Random walk

$$n'(t) = \int_0^t F(t')\,dt'$$

stochastic force

$$\langle F(t)\rangle = 0$$

$$\langle F(t)F(t+\Delta t)\rangle = \left\langle F^2\right\rangle e^{-\Delta t/\tau_c}$$

$$\left\langle \left(\lambda(t) - \lambda_0(t)\right)^2 \right\rangle$$

$$= \iiiint_0^{t,t',t,t'''} F(t'')F(t'''')\,dt''''\,dt'''\,dt''\,dt'$$

$$= \left\langle F^2\right\rangle \left(-2\tau^4 + \left(2\tau^3 t + 2\tau^4 + \tau^2 t^2\right) e^{-t/\tau} + \frac{1}{3}\tau t^3\right)$$

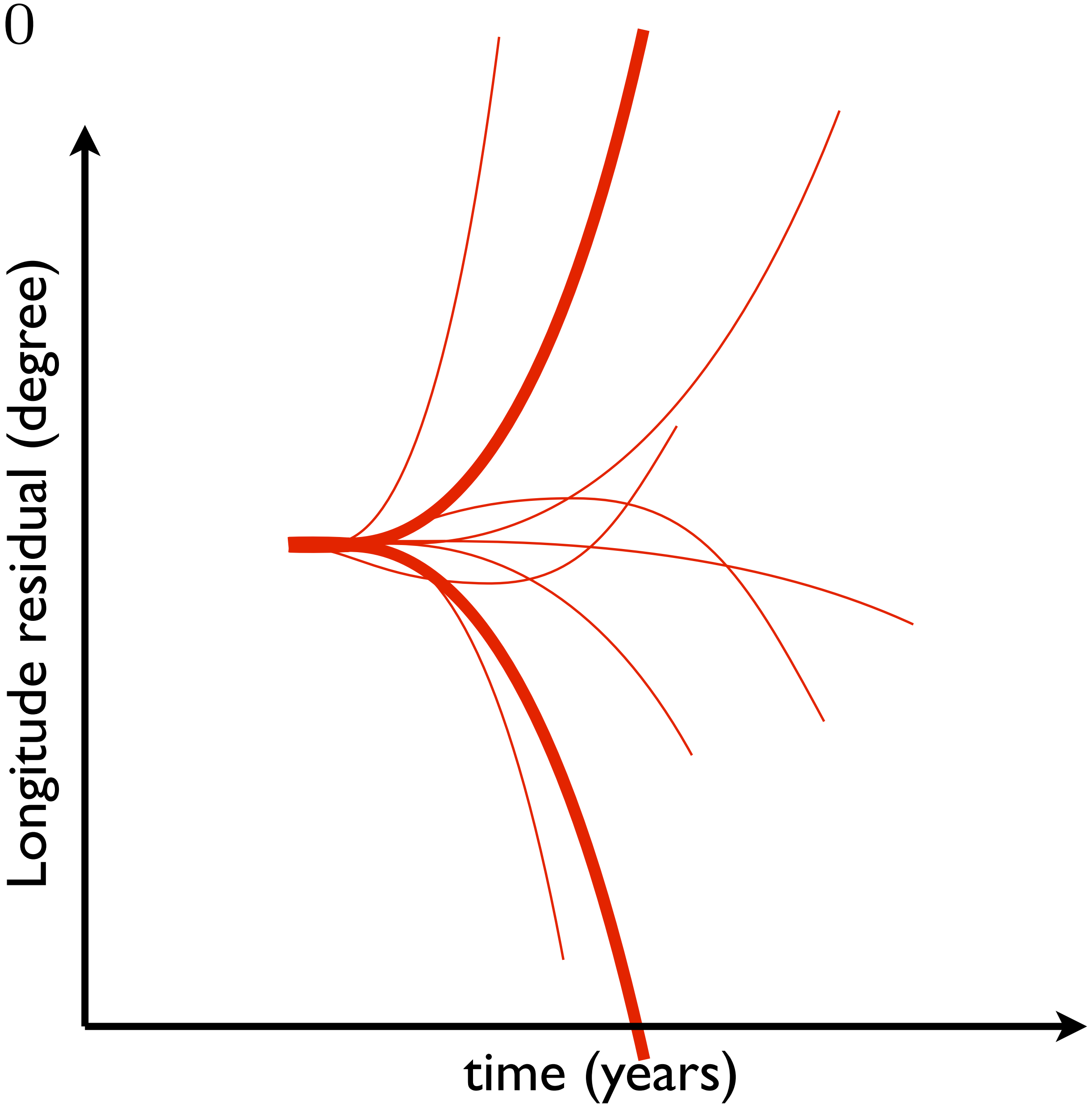
Random walk

$$n'(t) = \int_0^t F(t')\, dt'$$

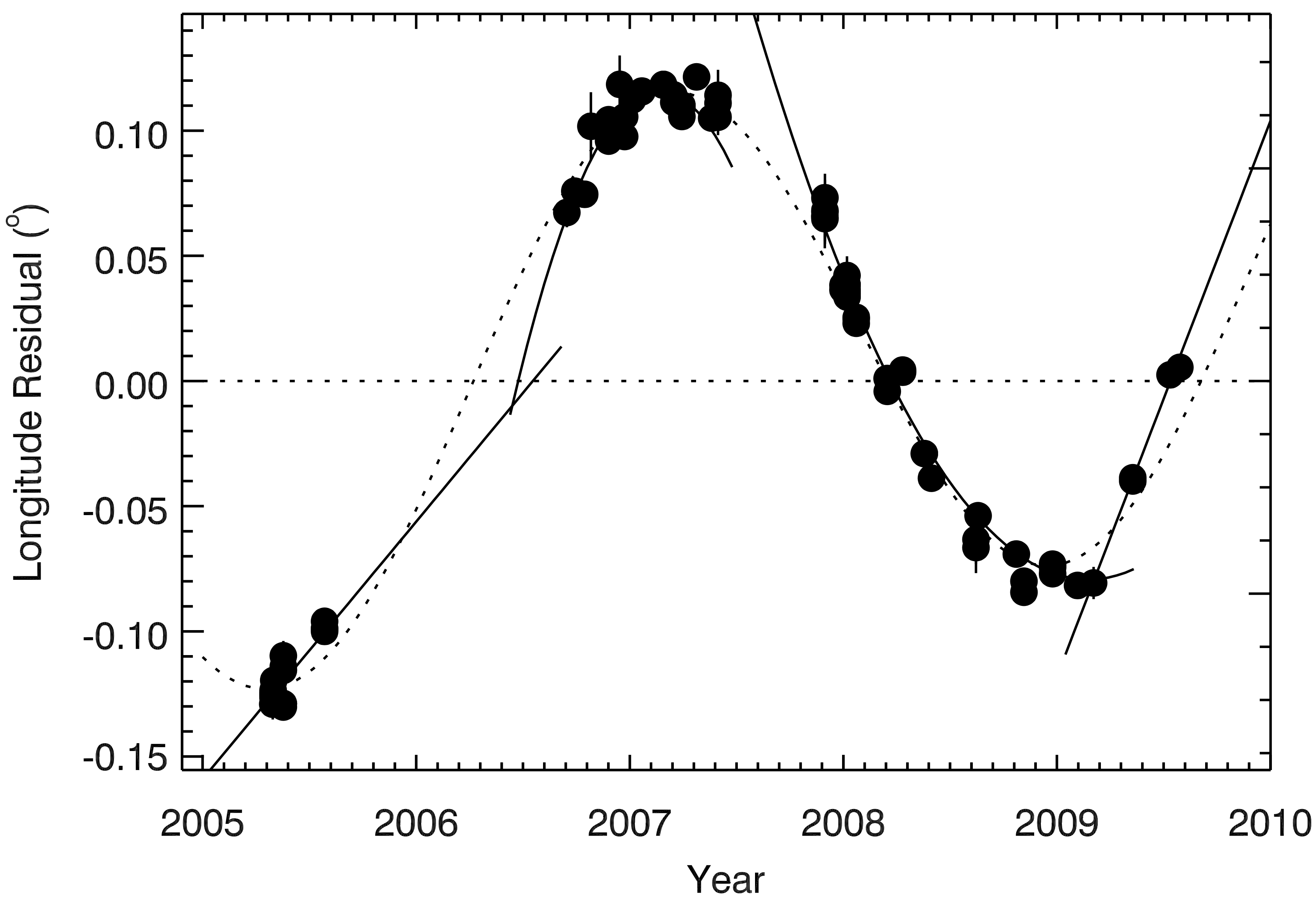
$$|\lambda(t) - \lambda_0(t)|$$

$$= \sqrt{\frac{\langle F^2 \rangle}{\tau}} t^{3/2}$$

On average!



Observational evidence of non-Keplerian motion



Tiscareno et al. 2010

Symplectic integrators

Observations

Possible explanations

Resonance with a moon

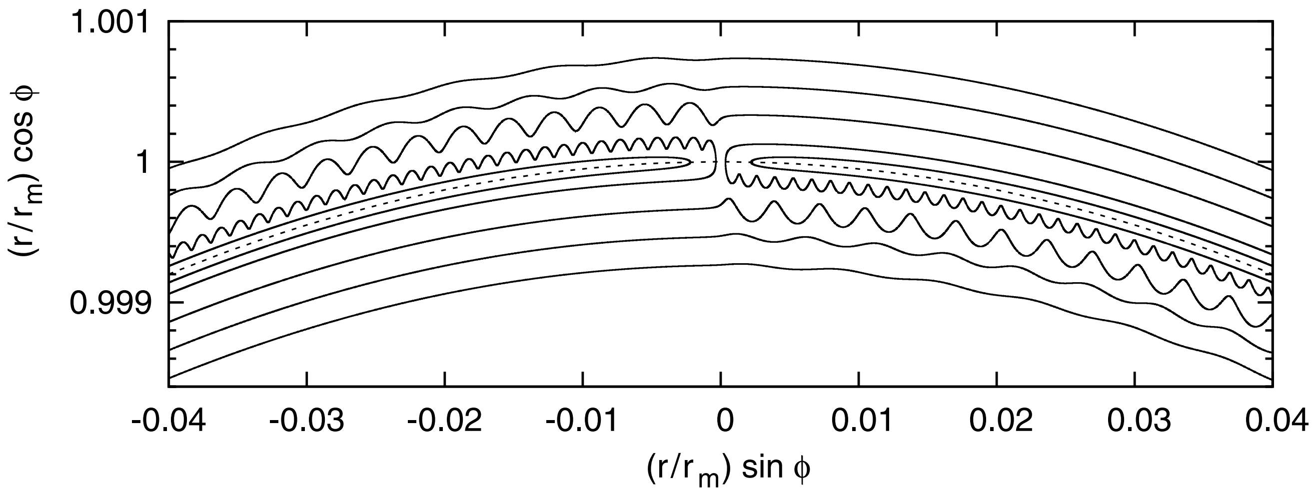
PRO

- Produces sine-shaped residual longitude
- Amplitude is a free parameter

CONTRA

- No resonance found
- Cannot fully explain shape of observations
- Other moonlets seem to migrate as well

Modified Type I Migration



- Due to curvature (would be zero in shearing sheet)
- Similar to planetary migration in a gas disk
- No gas pressure
- Migration rate can be calculated analytically

$$\frac{dr_m}{dt} = -35.6\,\frac{\Sigma {r_m}^2}{M}\left(\frac{m}{M}\right)^{1/3}\; r_m\Omega.$$

Crida et al. 2010

Modified Type I Migration

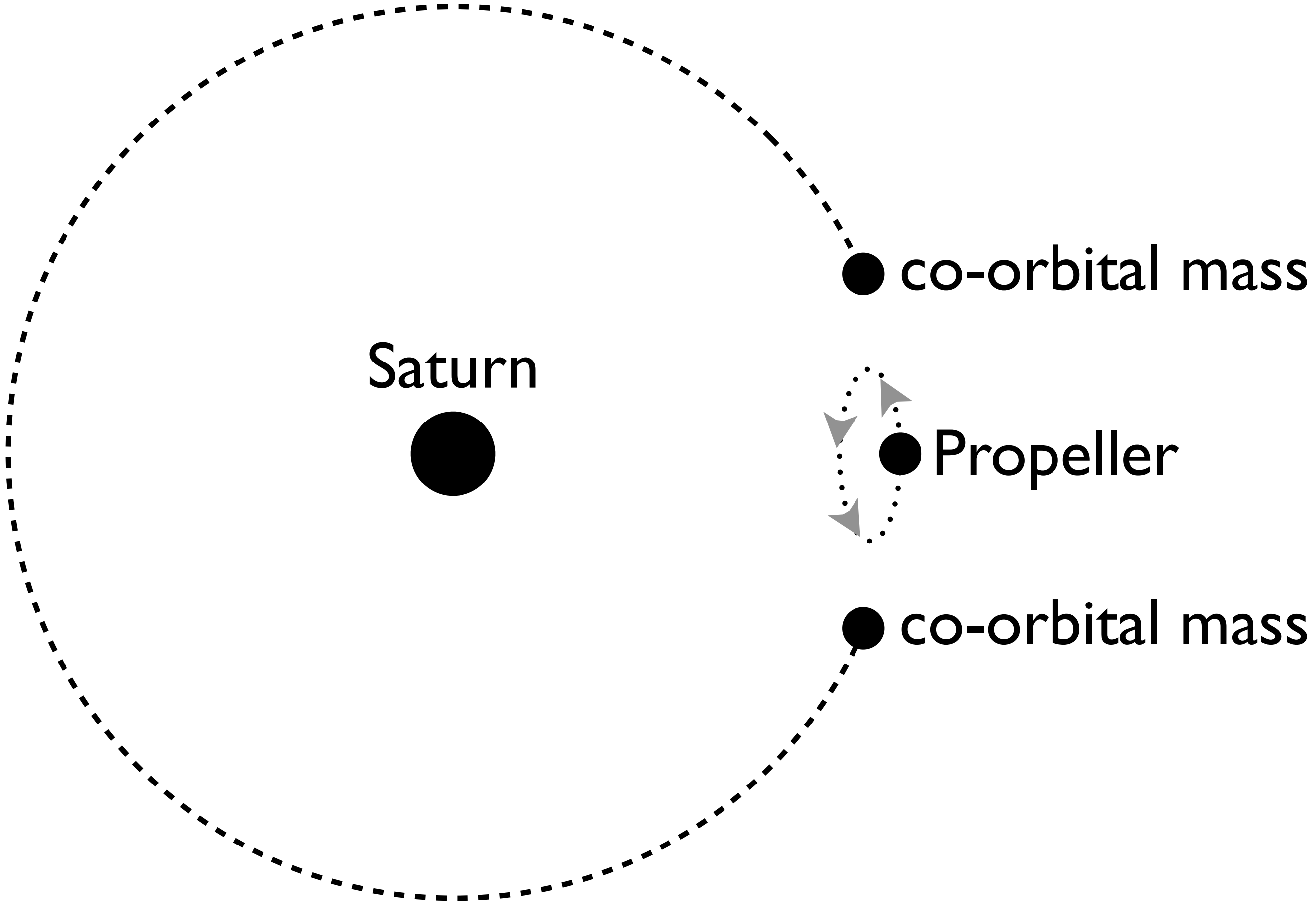
PRO

- Robust
- Would be a direct observation of type I migration

CONTRA

- Tiny migration rate ~20 cm/year
- Cannot explain shape of observations

Frog resonance



Frog resonance

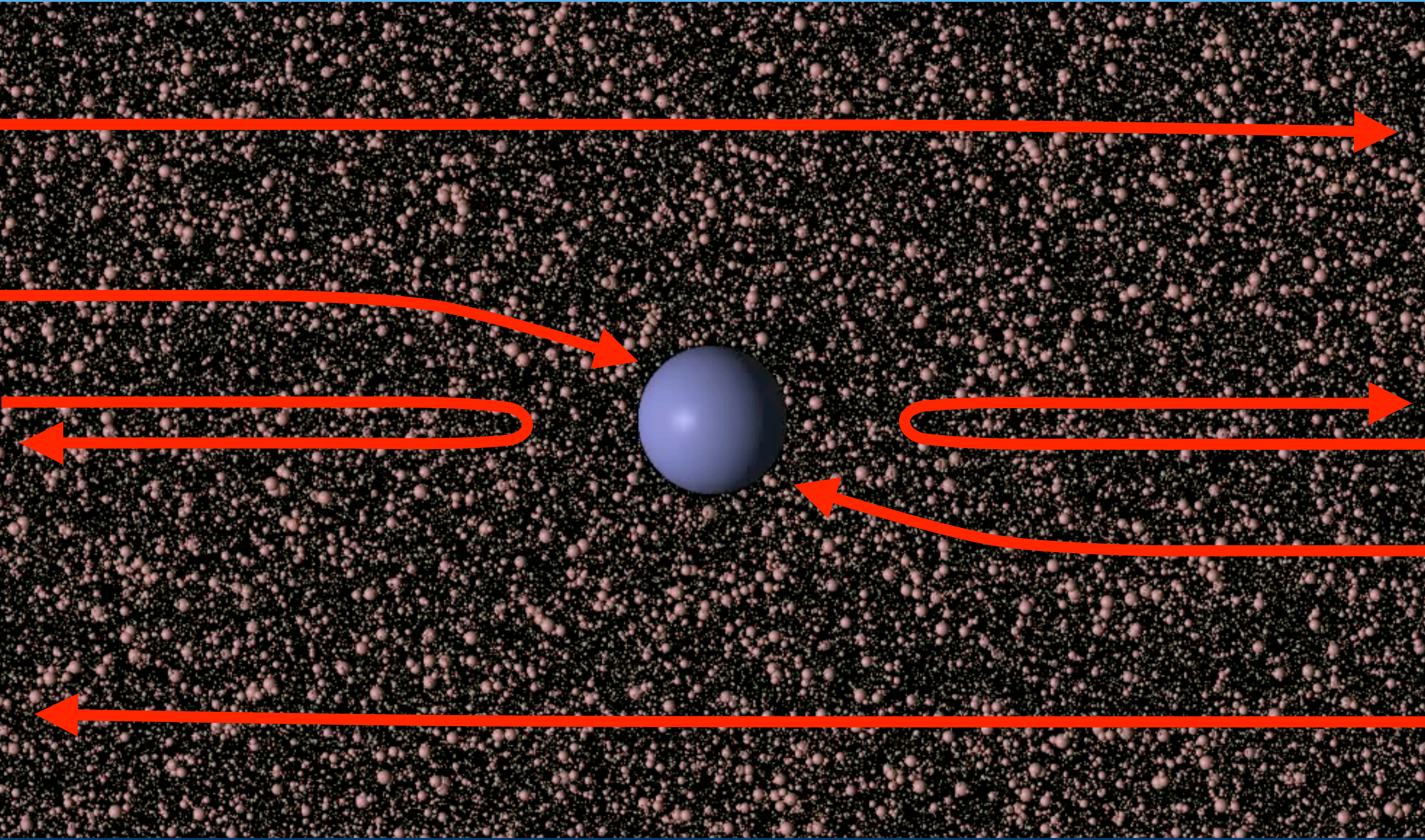
PRO

- Predicts largest period very well
- Amplitude is a free parameter

CONTRA

- Unclear if density distribution is like in the toy model (see Eugene's ISIMA project)
- Cannot fully explain shape of observations

Random Walk



Rein & Papaloizou 2010, Crida et al 2010

Two different approaches

Analytic model

Describing evolution in a statistical manner

Partly based on Rein & Papaloizou 2009

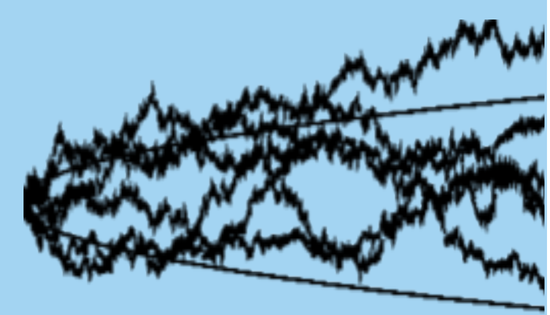
$$\Delta a = \sqrt{4\frac{Dt}{n^2}}$$

$$\Delta e = \sqrt{2.5\frac{\gamma Dt}{n^2 a^2}}$$

N-body simulations

Measuring random forces or integrating moonlet directly

Crida et al 2010, Rein & Papaloizou 2010



Rein & Papaloizou 2010, Crida et al 2010

Effects contributing to the eccentricity evolution

- Laminar collisions
- Particles colliding
- Laminar horseshoe
- Laminar circulating
- Particles circulating
- Clumps circulating

→ **Equilibrium eccentricity**

Damping | Excitation

... semi-major axis evolution

Particles colliding

Particles horseshoe

Particles circulating

Clumps circulating

Damping

Excitation

→

Random walk in semi-major axis

+Net "Type I" migration

Rein & Papaloizou 2010, Crida et al 2010

Random Walk

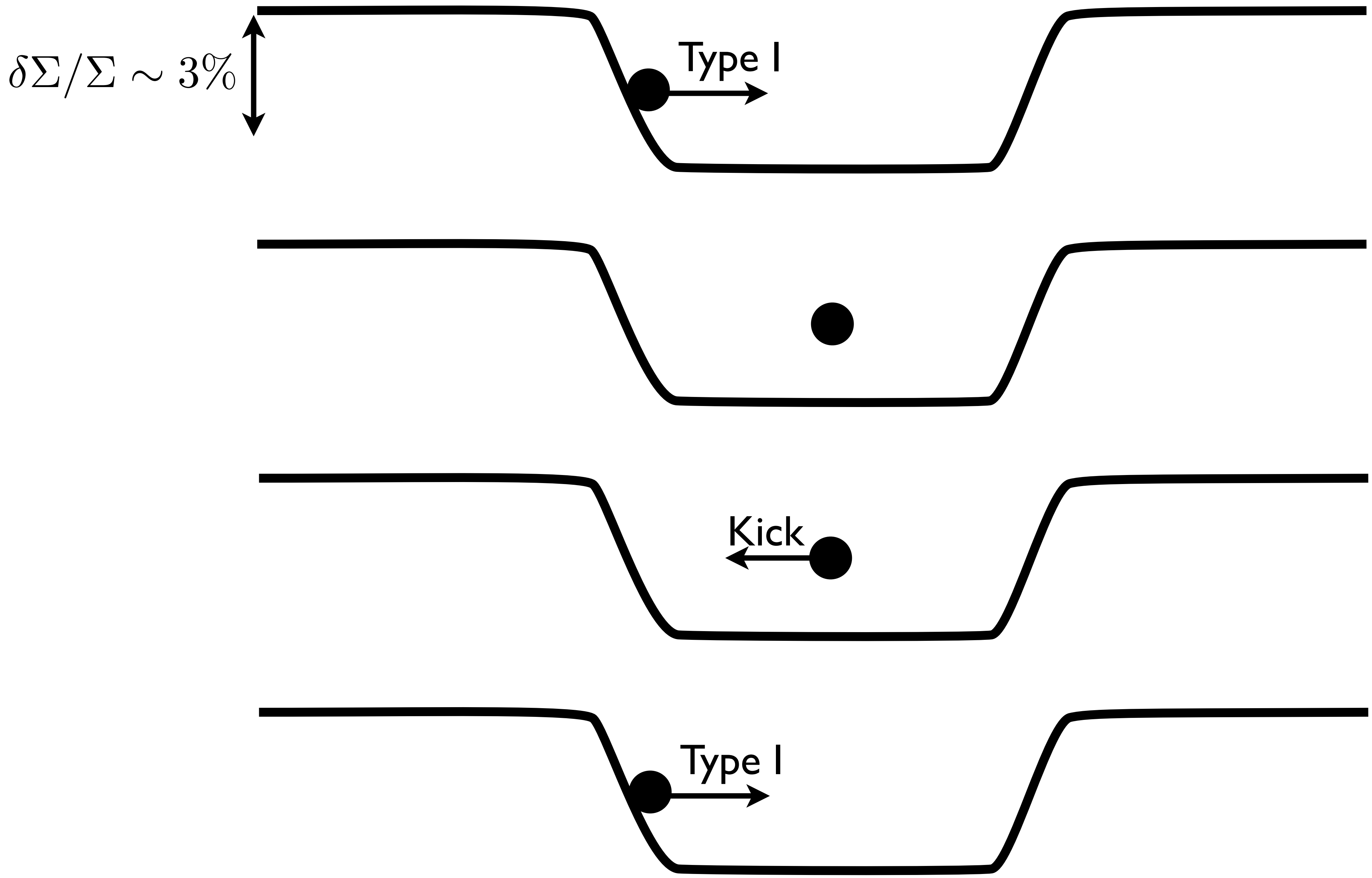
PRO

- Can explain the shape of the observations very well

CONTRA

- Has only been tested numerically for small moonlets (ISIMA project with Shangfei)
- No metric to test how good it matches the observations

Hybrid Type I Migration / Stochastic Kicks



Tiscareno (in prep)

Hybrid Type I Migration / Stochastic Kicks

PRO

- Can explain all observations very well

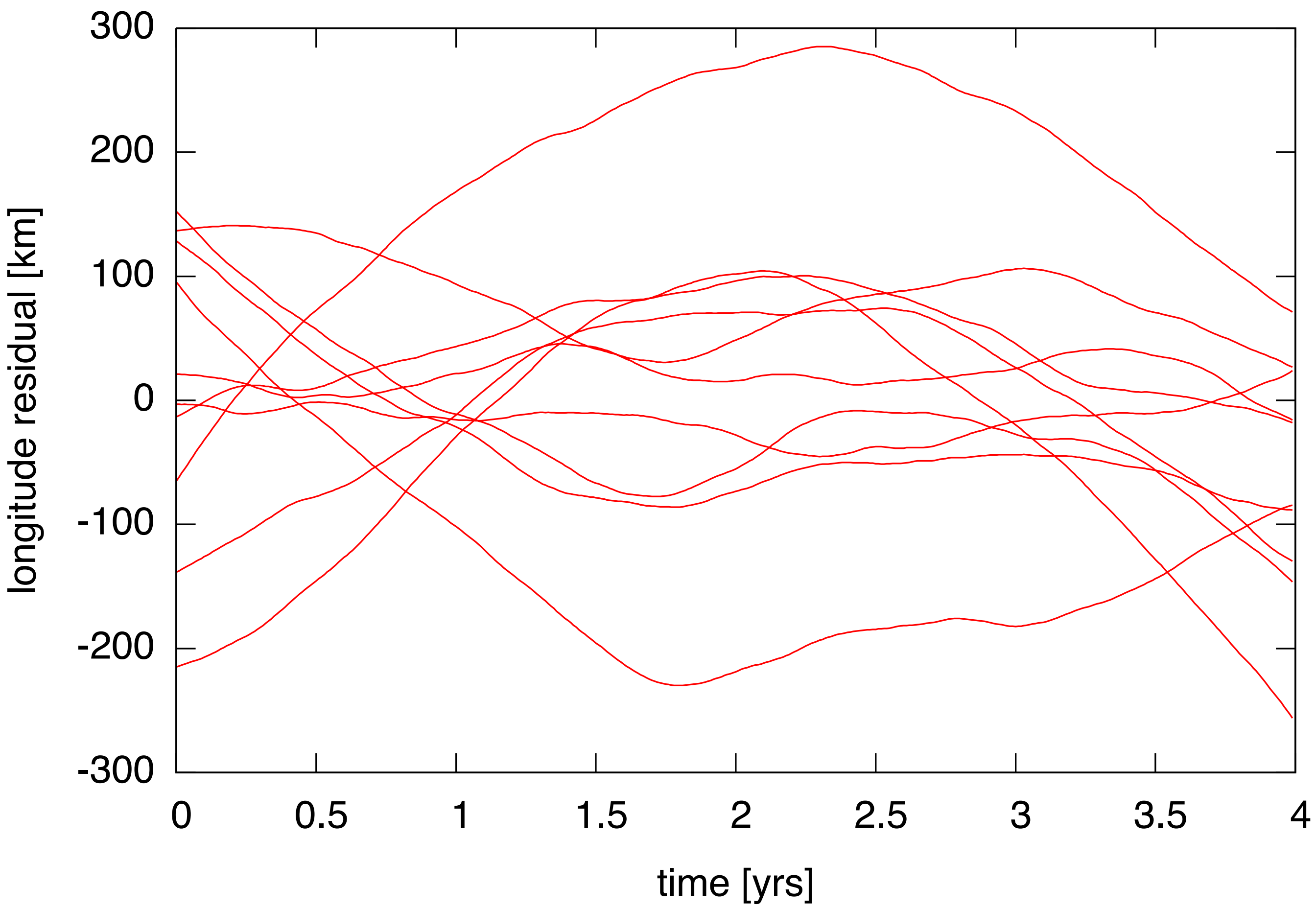
CONTRA

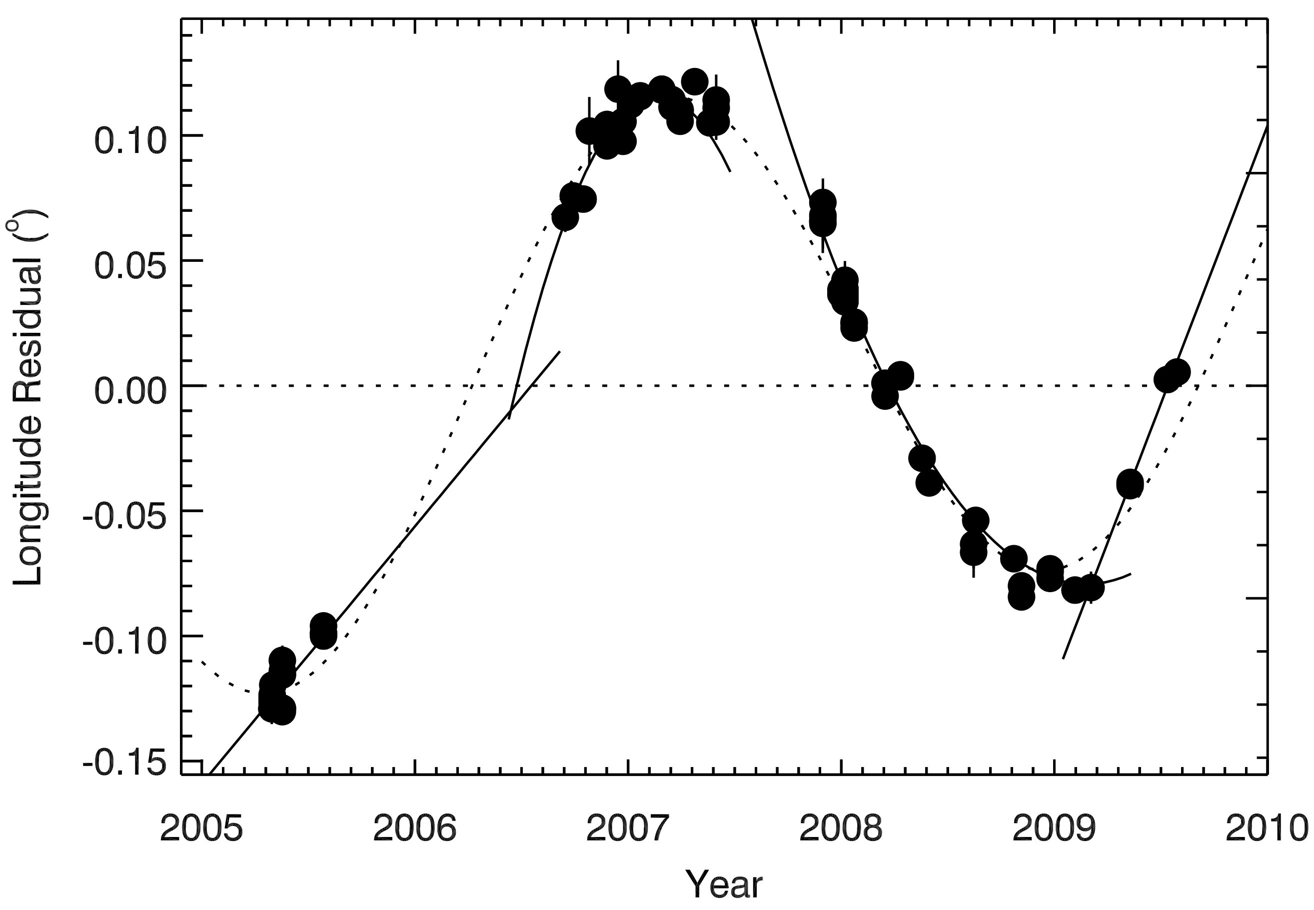
- Many free parameters: surface density profile, kicks

- Needs large kicks (maybe not)

Tiscareno (in prep)

Need a metric





Conclusions
Part II

Conclusions

Moonlets in Saturn's rings

Small scale version of the proto-planetary disc
Dynamical evolution can be directly observed
5 different explanations
Might lead to independent age estimate of the ring system

- Modified Type I Migration
- Random Walk
- Frog Resonance
- Hybrid Migration/ Random Walk
- Resonance with a moon

Thank you!