

# Simulating planetary systems

Newton's law of gravity, machine learning,  
and everything in-between



Hanno Rein, University of Toronto

Image credit: NASA/SOFIA/Lynette Cook

# Agenda

**Early work on Solar System Dynamics**

**Chaos**

**Modern numerical methods**

**Machine learning**

**REBOUND, REBOUNDx, ASSIST**

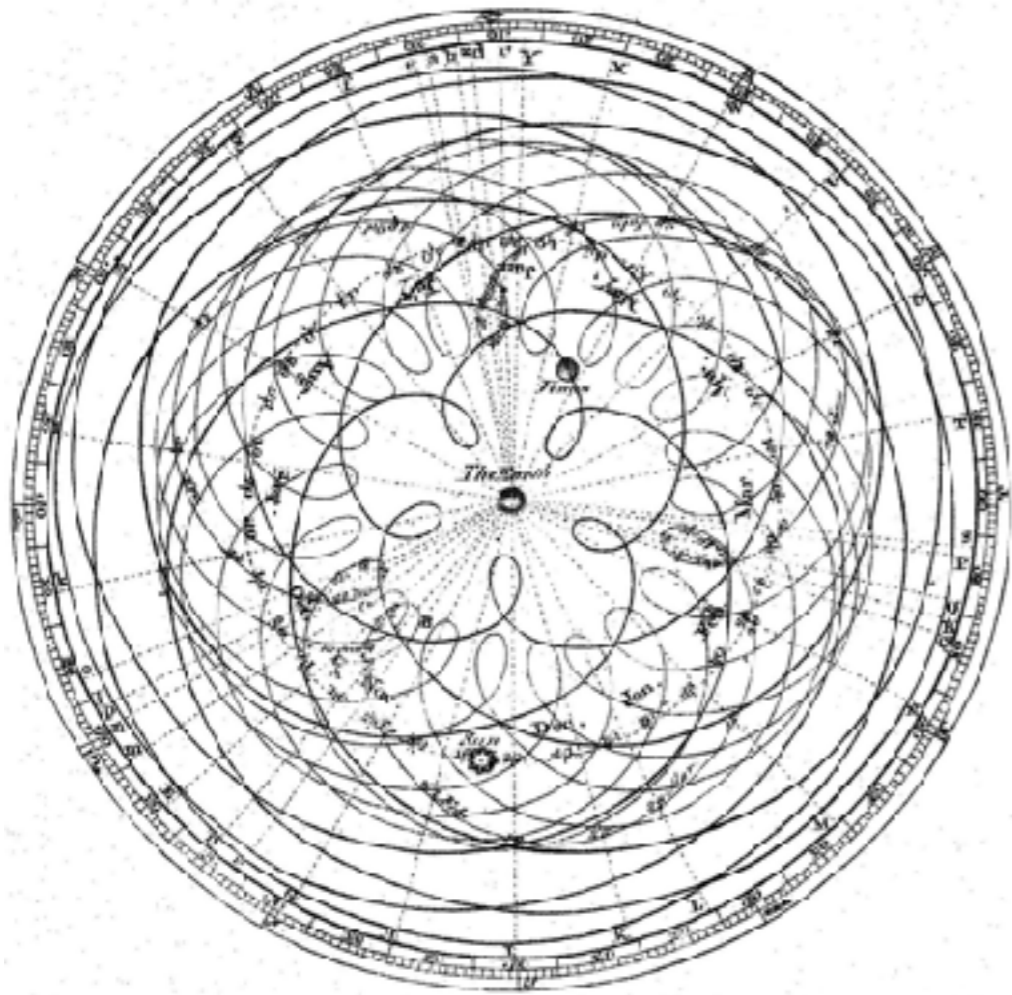
**Light pollution**

# Early work on Solar System Dynamics

# Newton (1687)

$$\ddot{\mathbf{r}}_i = \sum_{\substack{j=1 \\ j \neq i}}^N m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$

# Newton (Opticks 1717, 1730)



For while comets move in very excentrick orbs in all manner of positions, blind fate could never make all the planets move one and the same way in orbs concentrick, some inconsiderable irregularities excepted, which may have risen from the **mutual actions** of comets and planets upon one another, and which will be apt to increase, **till this system wants a reformation.**

# Evidence for irregularity/instability



Ptolemy

On March 1st, 228 BC, at 4:23 am, mean Paris time, Saturn was observed two fingers under Gamma in Virgo.

+

Observations from 1590 and 1650.

=

Six million years ago Jupiter and Saturn were at the same distance from the Sun.

Demo with REBOUND

# Laplace-Lagrange Secular Dynamics



Average over  
short time scales



Perturbation theory



# Explanations for the irregularities?



Euler was twice awarded a prize in 1748 and 1752 related to this problem by the Paris Academy of Sciences.

Lagrange thought that Euler's calculations were wrong and did his own.

# Laplace (1776)



Mr. Euler, in his second piece on the irregularities of Jupiter and Saturn, find it **equal for both these planets**. According to Mr. de Lagrange, on the contrary, [...] it is very **different for these two bodies**. [...] I have some reasons to believe, however, that the formula is still not accurate. The one which I obtain is quite different. [...] by substituting these values in the formula of the secular equation, **I found absolutely zero**, from which I conclude the alteration of the mean motion of Jupiter, if it exists, does not result from the action of Saturn.

# Lagrange (in a letter to d'Alembert, 1775)



I am ready to give a **complete theory for the variations of the elements of the planets** under their mutual action. That Mr. de la Place did on this subject I liked, and I flatter myself that he will not be offended if I do not hold the kind of promise that I made to completely abandon this subject to him; I could not resist to the desire to look into it again, but I am no less charmed that he is also working on it on his side; I am even very eager to read his subsequent research on this topic, but I do ask him not to send me any manuscript and send them to me only in printed form.

# Fundamental modes, eigenfrequencies

	S <sub>1</sub>	S <sub>2</sub>	S <sub>3</sub>	S <sub>4</sub>	S <sub>5</sub>	S <sub>6</sub>	S <sub>7</sub>	S <sub>8</sub>
Lagrange (1774)	5.98	6.31	19.80	18.31	0	25.34	-	-
Brown & Rein (2019)	5.59	7.05	18.84	17.74	0	26.35	2.99	0.69

Note: No semi-major axis changes to first and also second order (Poisson, Haretu and Poincaré) in the expansion.

This still contradicts Ptolemy's observations from antiquity.

Demo with REBOUND

# Laplace (1785)

Simple energy argument implies:

$$\frac{m_J}{a_J} + \frac{m_S}{a_S} = \text{const}$$

Thus, can be confident that the change in orbits must be due to mutual interactions.

He's also shown, no secular terms. Hence must be short period.

Near 5:2 mean motion resonance. Period of 900 years.

# Secular Dynamics



Average over  
short time scales



Perturbation theory



5:2 near MMR

# Secular Dynamics



Average over  
short time scales

Accurate over  
> 1 million years



5:2 near MMR



**Chaos**

# Le Verrier (1840, 1841)



Follow up on the work of Lagrange and Laplace but to higher order.

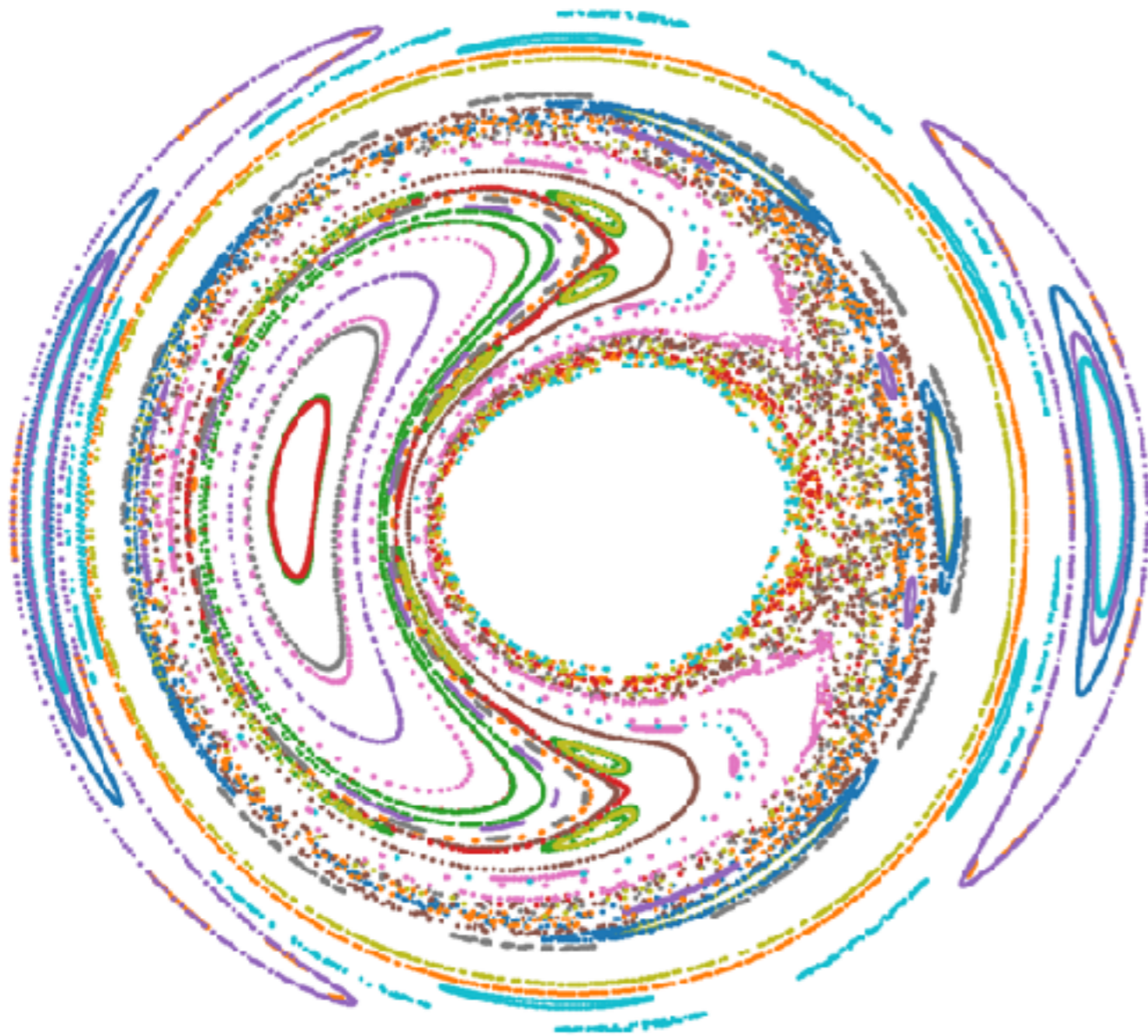
Small divisor problem:  
third order could be larger than  
second order terms

# Poincaré (1897)



The terms of these series, in fact, decrease first very quickly and then begin to grow, **but as the Astronomers' stop after the first terms of the series**, and well before these terms have stop to decrease, the approximation is sufficient for the practical use. The divergence of these expansions would have some disadvantages only if one wanted to use them to rigorously establish some specific results, as the **stability of the Solar System**.

# Kolmogorov (1954), Arnold (1963), Moser (1962)

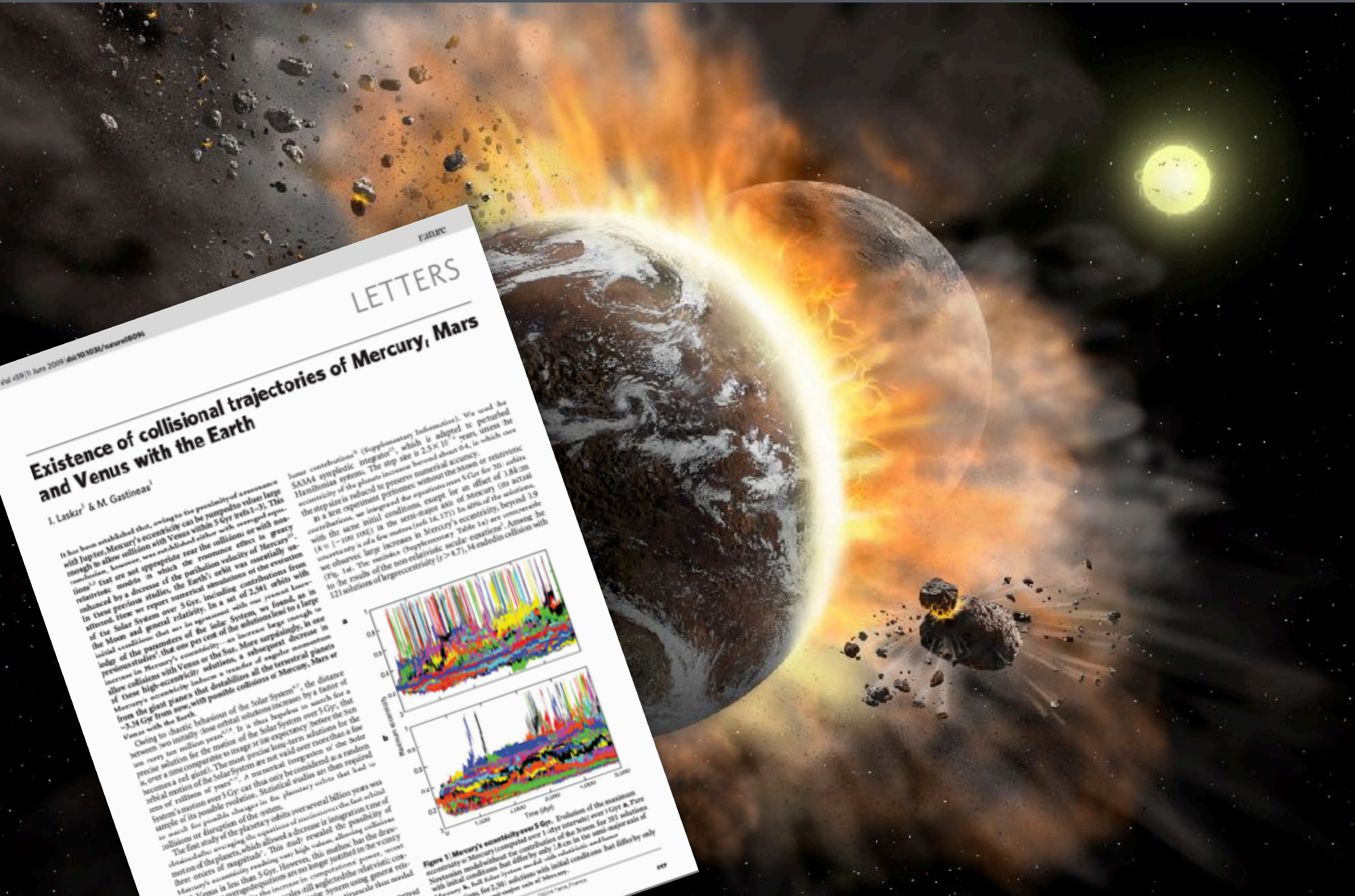


Kolmogorov showed that convergent perturbation series can exist.

Many subtleties (degeneracy, small masses, slow Arnold diffusion)

In short: expansions are not useful for determining the stability of our Solar System.

# Laskar (2009)



Vol 459 | 11 June 2009 | doi:10.1038/nature08094

nature

## LETTERS

### Existence of collisional trajectories of Mercury, Mars and Venus with the Earth

J. Laskar<sup>1</sup> & M. Gastineau<sup>1</sup>

It has been established that, owing to the proximity of a resonance with Jupiter, Mercury's eccentricity can be pumped to values large enough to allow collision with Venus within 5 Gyr (refs 1–3). This conclusion, however, was established either with averaged equations<sup>1,2</sup> that are not appropriate near the collision or with non-retrograde models in which the resonance effect is greatly enhanced by a decrease of the Earth's orbit as the evolution of the Solar System over 5 Gyr, including contributions from the Moon and general relativity. In a set of 2,581 orbits with initial conditions that are in agreement with our present knowledge of the parameters of the Solar System, we found, as in previous studies<sup>1</sup>, that one per cent of the solutions lead to a large increase in Mercury's eccentricity—an increase large enough to allow collisions with Venus or the Sun. More surprisingly, in one of these high-eccentricity solutions, a subsequent decrease in Mercury's eccentricity induces a transfer of angular momentum from the giant planets that destabilizes all the terrestrial planets Venus with the Earth.

Owing to chaotic behaviour of the Solar System<sup>4,5</sup>, the distance between two initially close orbital solutions increases by a factor of ten every ten million years<sup>6,7</sup>. It is thus hopeless in search for a precise solution for the motion of the Solar System over 5 Gyr, that is, over a time comparable to its age or life expectancy (before the Sun becomes a red giant). The most precise long-term solutions for the orbital motion of the Solar System are not valid over more than a few tens of millions of years<sup>8–10</sup>. A numerical integration of the Solar System's motion over 5 Gyr can thus only be considered as a random sample of its possible evolutions. Statistical studies are then required to search for possible changes in the planetary orbits that lead to collisions or disruption of the system.

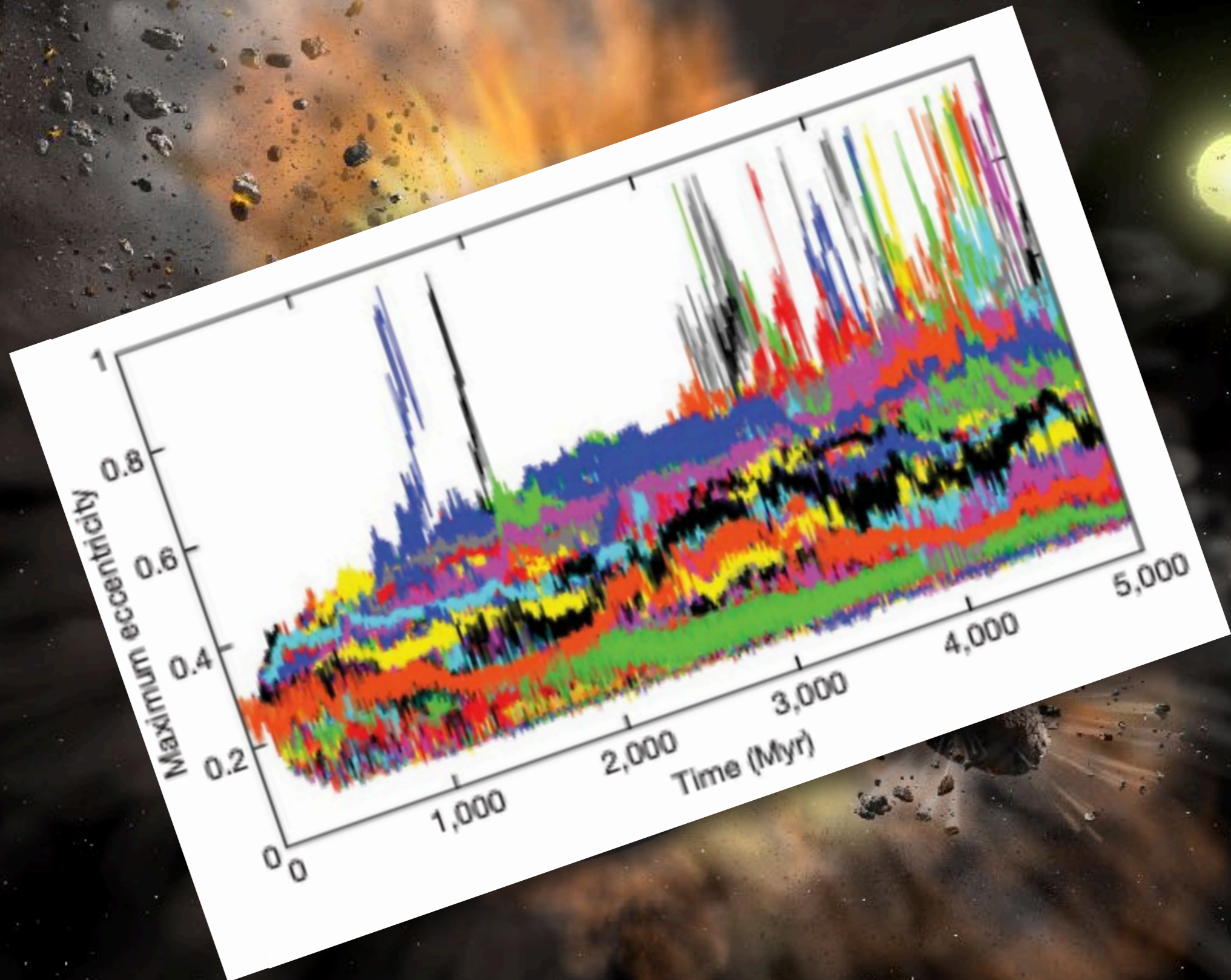
The first study of the planetary orbits over several billions years was obtained by averaging the equations of motion over the fast orbital motion of the planets, which allowed a decrease in integration time of three orders of magnitude<sup>11</sup>. This study revealed the possibility of Mercury's eccentricity reaching very high values, allowing collisions with Venus or the Sun. However, this method has the drawback that the increase in computational power, recent advances in computer architecture and the use of general relativity in the Solar System using general relativity<sup>12</sup> have made it possible to integrate the equations of motion at the full scale thus needed

lunar contributions<sup>13</sup> (Supplementary Information). We used the SAAS44 symplectic integrator<sup>14</sup>, which is adapted to perturbed Hamiltonian systems. The step size is  $2.5 \times 10^{-2}$  years, unless the step size is reduced to preserve numerical accuracy. In a first experiment performed without the Moon or relativistic contributions, we integrated the equations from 5 Gyr for 201 orbits with the same initial conditions, except for an offset of 1.8 km in the semi-major axis of Mercury (its actual uncertainty is of a few metres (ref 16, 17)). In 20% of the solutions, we observed large increases in Mercury's eccentricity, beyond 0.9 (Fig. 1a). The statistics (Supplementary Table 1a) are comparable to the results of the non-relativistic secular equations<sup>15</sup>. Among the 121 solutions of large eccentricity ( $e > 0.7$ ), 14 ended in collision with

**Figure 1** Mercury's eccentricity over 5 Gyr. Evolution of the maximum eccentricity of Mercury (computed over 1-Myr intervals) over 5 Gyr. **a**, Pareto Newtonian model without the contribution of the Moon for 201 solutions with initial conditions that differ by only 1.8 cm in the semi-major axis of Mercury. **b**, Full Solar System model with relativistic and lunar contributions, for 250 solutions with initial conditions that differ by only 1.8 cm in the semi-major axis of Mercury.

119

# Laskar (2009)



# Modern numerical methods

# Newton (1687)

$$\ddot{\mathbf{r}}_i = \sum_{\substack{j=1 \\ j \neq i}}^N m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$



# How to solve N-body ODE

Brute force approach  
(IAS15)

Physical approach  
(WHFast/EOS)



# Leap frog integrator

$$H = T + U$$

**A**

$$T = \sum_{i=0}^{N-1} T_i$$

**B**

$$U = \sum_{i=0}^{N-1} \sum_{j=i+1}^{N-1} U_{ij}$$

Both solutions are trivial!

$$\dot{r}_i = v_i$$

$$\dot{v}_i = \sum_{j \neq i} a_{ij}$$

# Wisdom-Holman integrator

$$H = A + B$$

**A**

$$\sum_{i=0}^{N-1} T_i + \sum_{i=1}^{N-1} U_{i0}$$

Dominant part of motion

**B**

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} U_{ij}$$

Perturbation

Solution for B is still trivial.

Solution for A is more complicated. We need a “Kepler solver”.

# Embedded Operator Splitting Method (EOS)

$$H = A + B$$

$$\mathbf{A} \quad \sum_{i=0}^{N-1} T_i + \sum_{i=1}^{N-1} U_{i0}$$

$$\mathbf{B} \quad \sum_{i=1}^{N-1} \sum_{j=i+1}^{N-1} U_{ij}$$

$$A = A_1 + A_2$$

$$\mathbf{A}_1 \quad \sum_{i=0}^{N-1} T_i$$

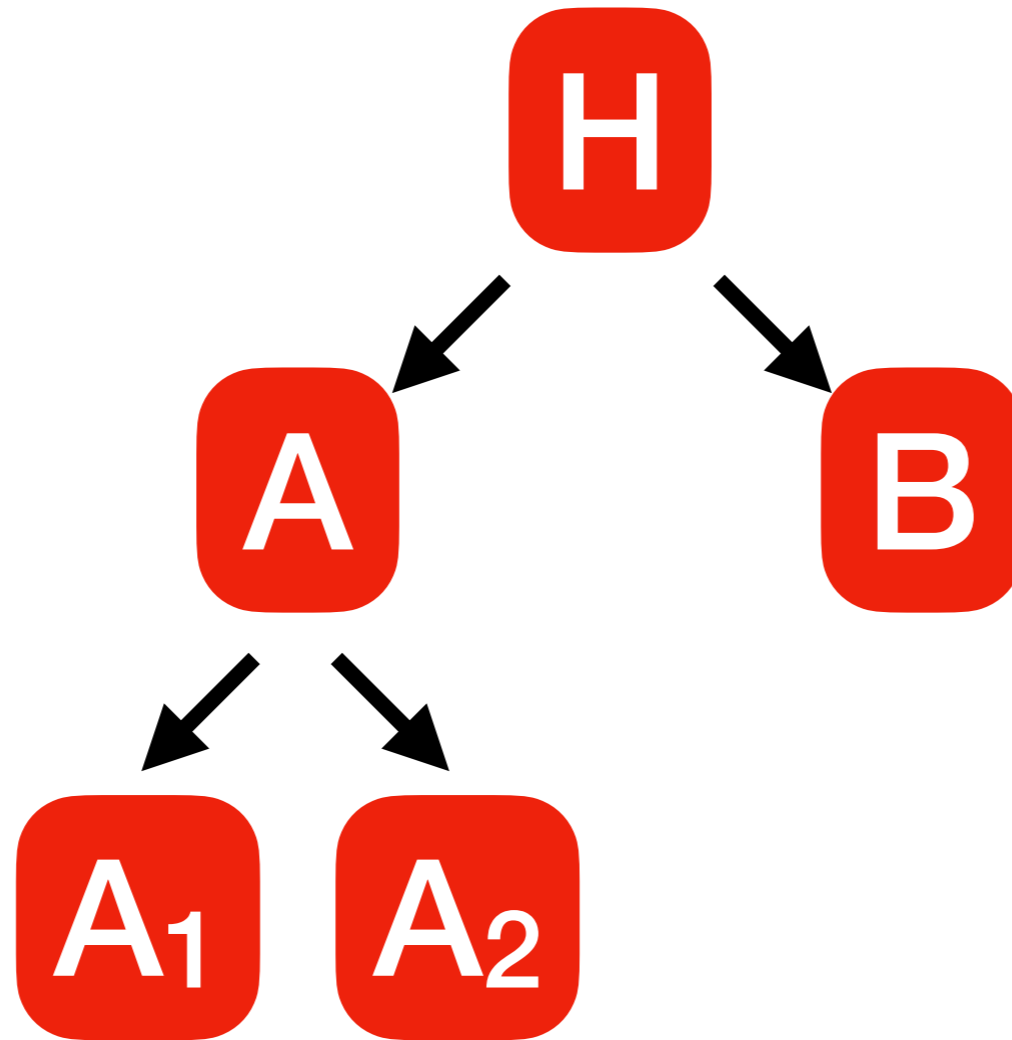
$$\mathbf{A}_2 \quad \sum_{i=1}^{N-1} U_{i0}$$

# A lot of choice

Full Hamiltonian

1st splitting

2nd splitting

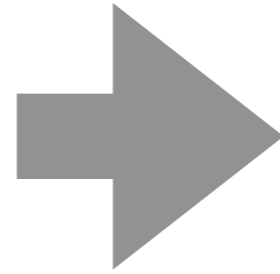


At each splitting, can choose:

- How to split Hamiltonian into two parts
- Which splitting method to use
- Timestep

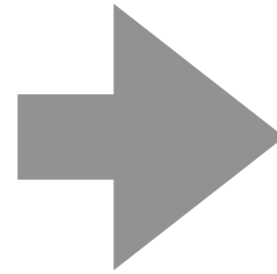
# EOS methods are extremely flexible

splitting into  $T + U$



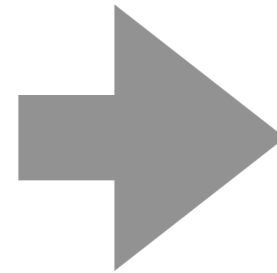
leap frog,  
higher order leap frog

splitting into Keplerian  
motion + perturbations



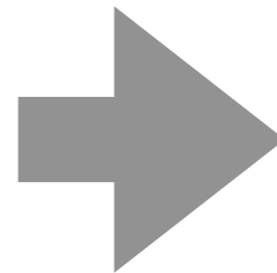
Wisdom Holman integrator,  
higher order generalizations

splitting into near and far  
interactions



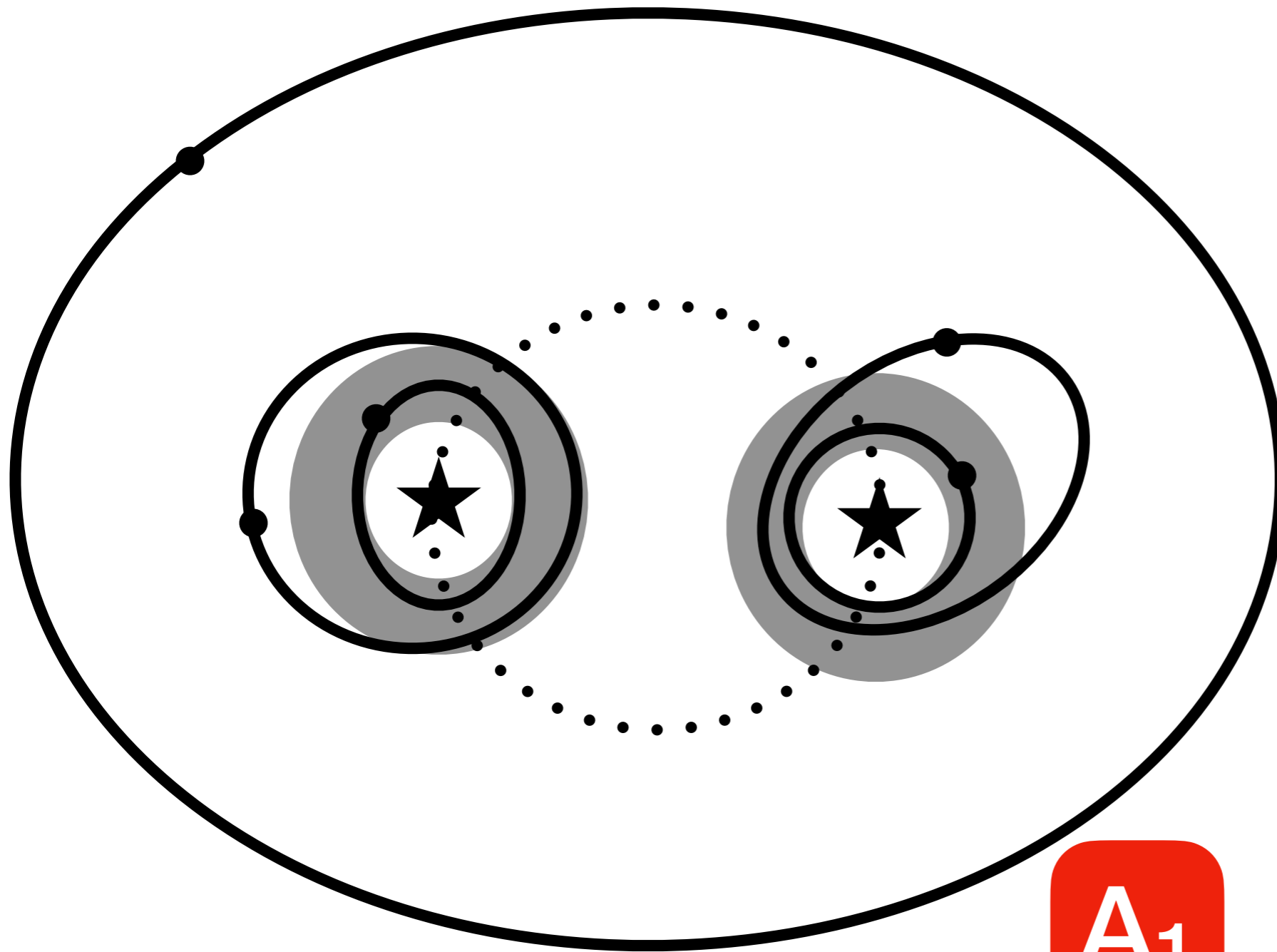
Hybrid symplectic  
integrators, Mercury

splitting into many different  
“shells”



SYMBA

# Example: complicated hierarchical systems



**B**

planet-planet interactions  
(kick)

**A<sub>1</sub>**

kinetic term (drift)

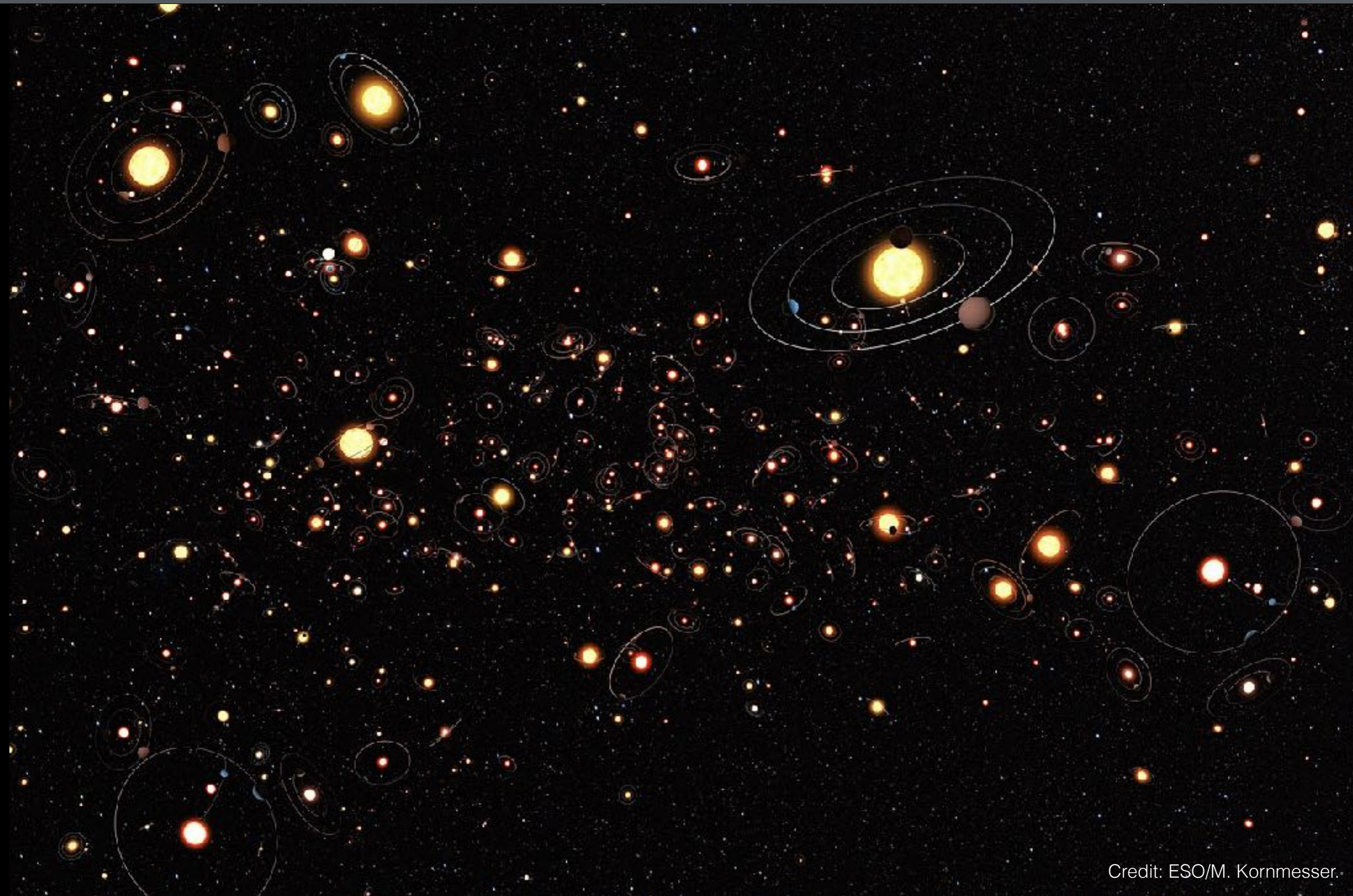
**A<sub>2</sub>**

planet-star and star-  
star interactions (kick)

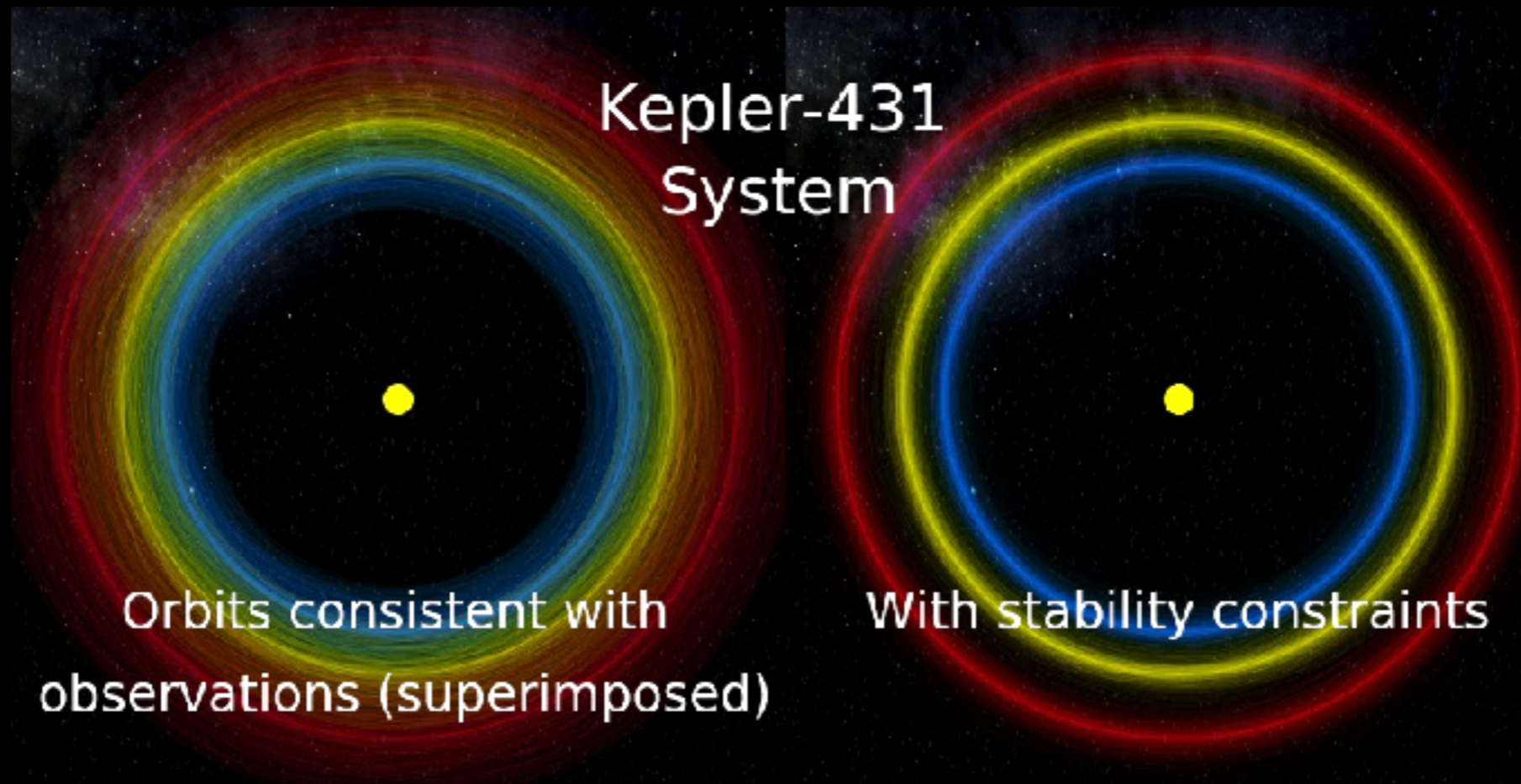


# Predicting the stability of planetary systems with machine learning

# Planets everywhere

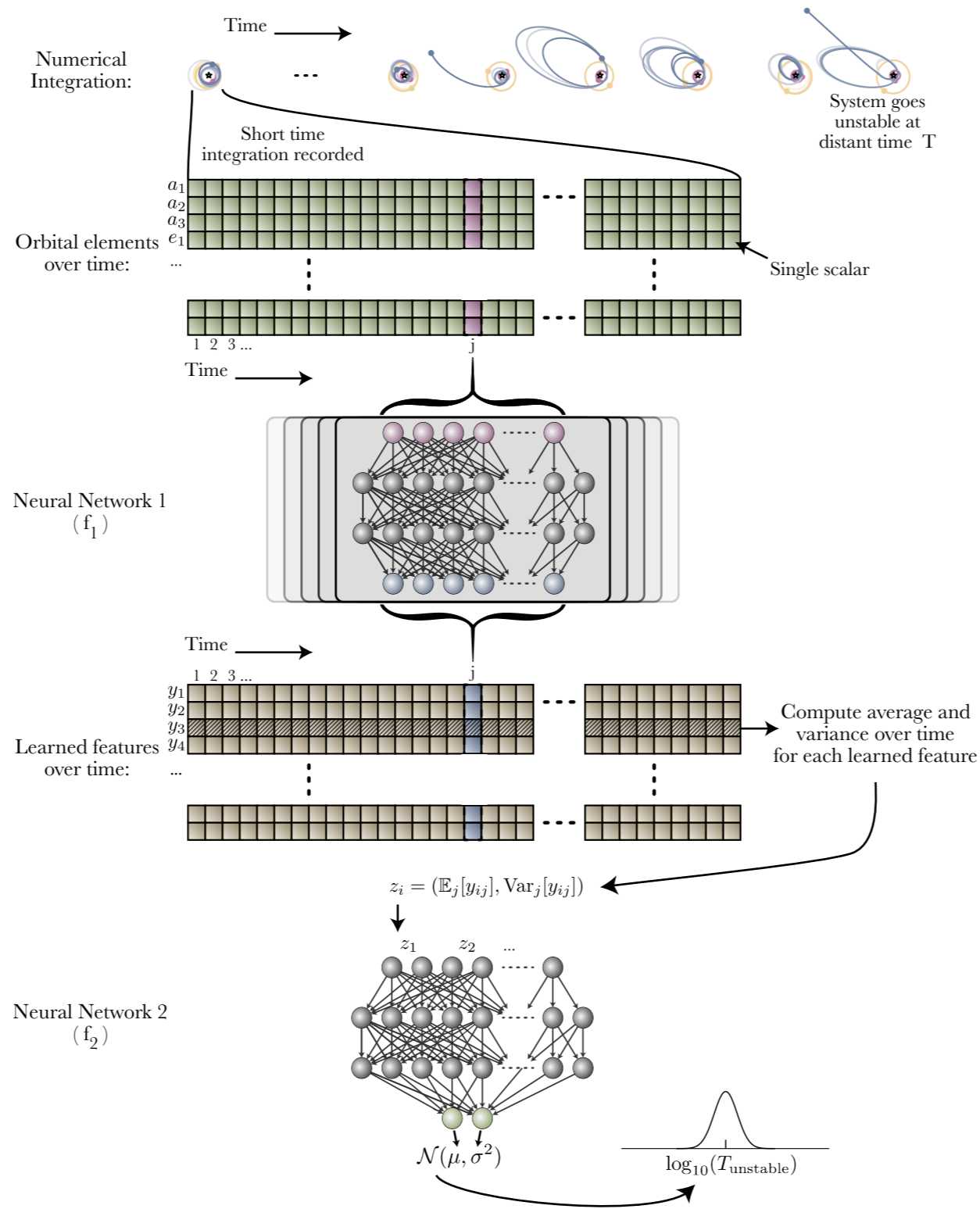


# Constrain orbital parameters

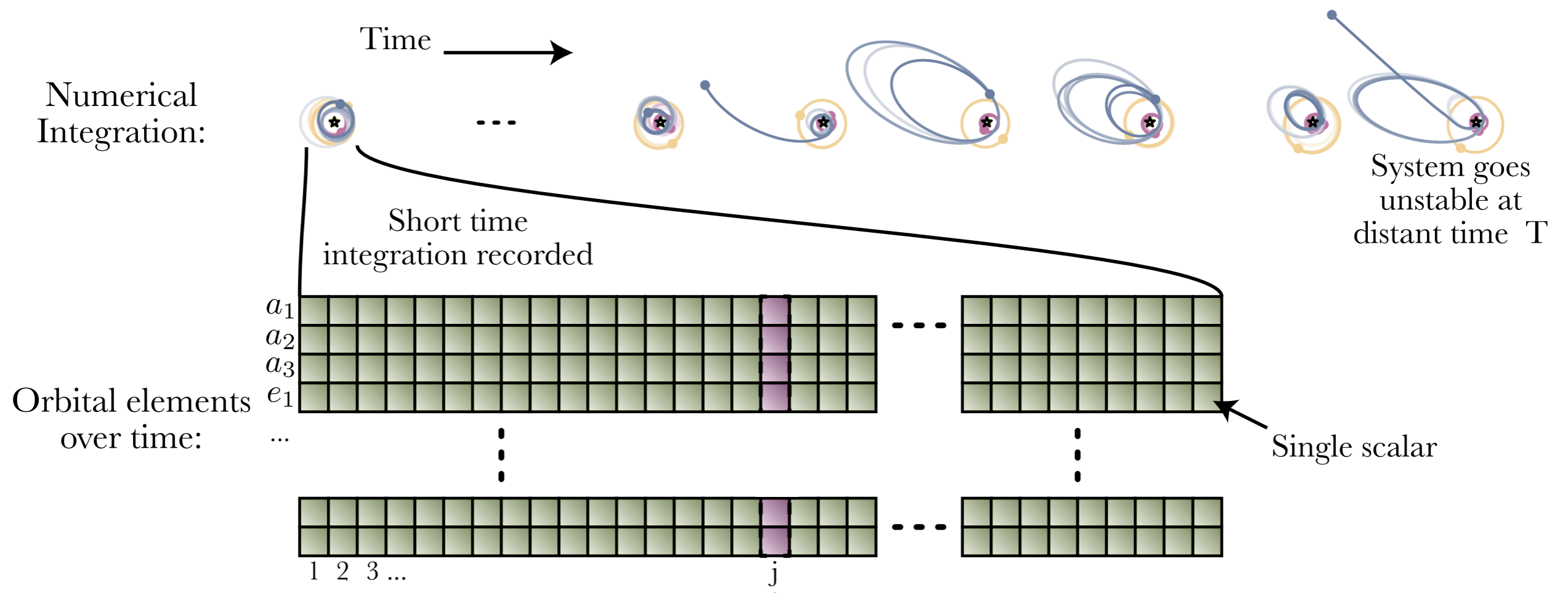


- ▶ This is a hard problem!
- ▶ Machine learning models can help solve this problem.
- ▶ Doesn't have to be a black box! Can be part of a Bayesian analysis.

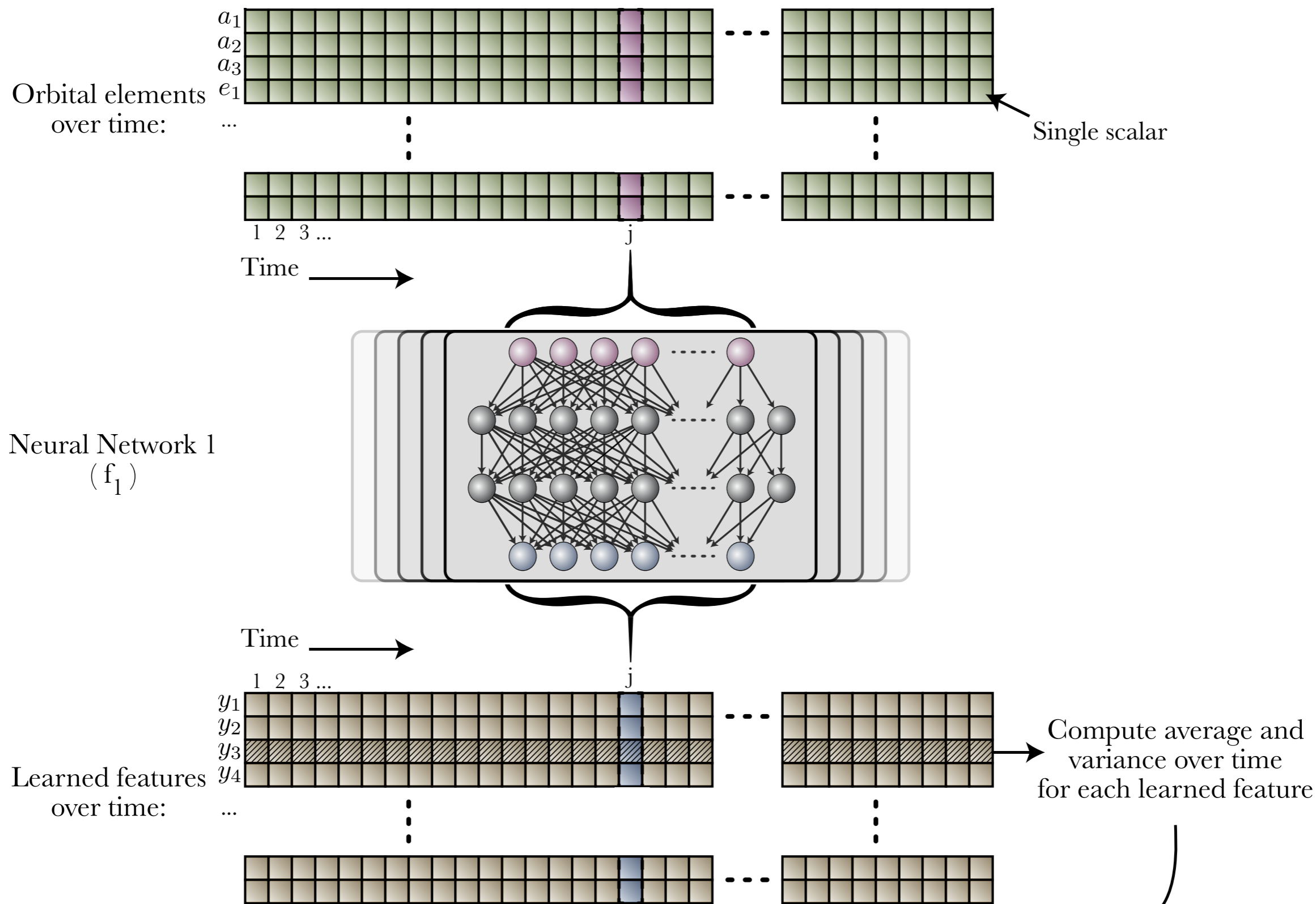
# Bayesian Neural Network



# Step 1: Short integration

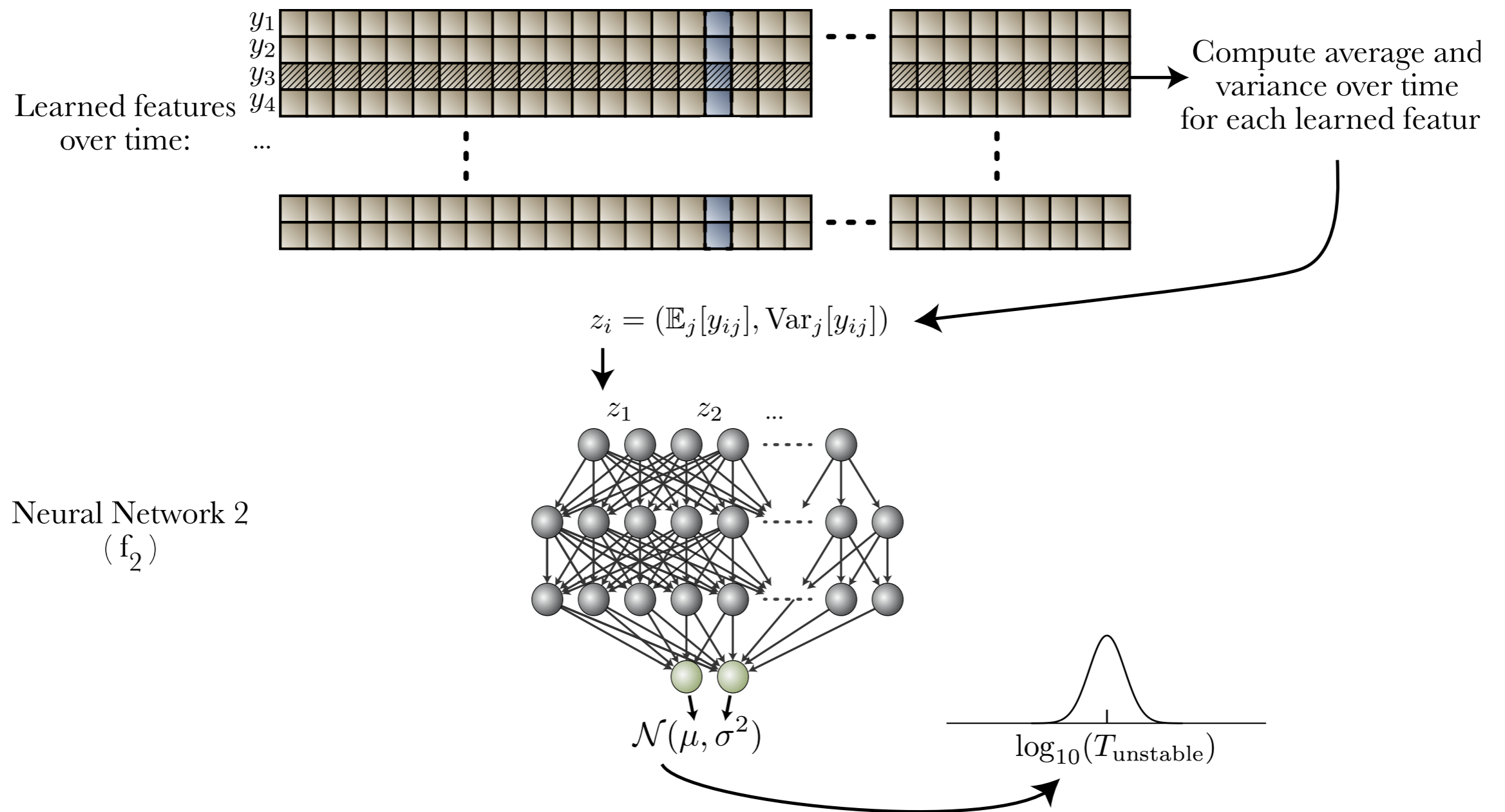


# Step 2: Learning features

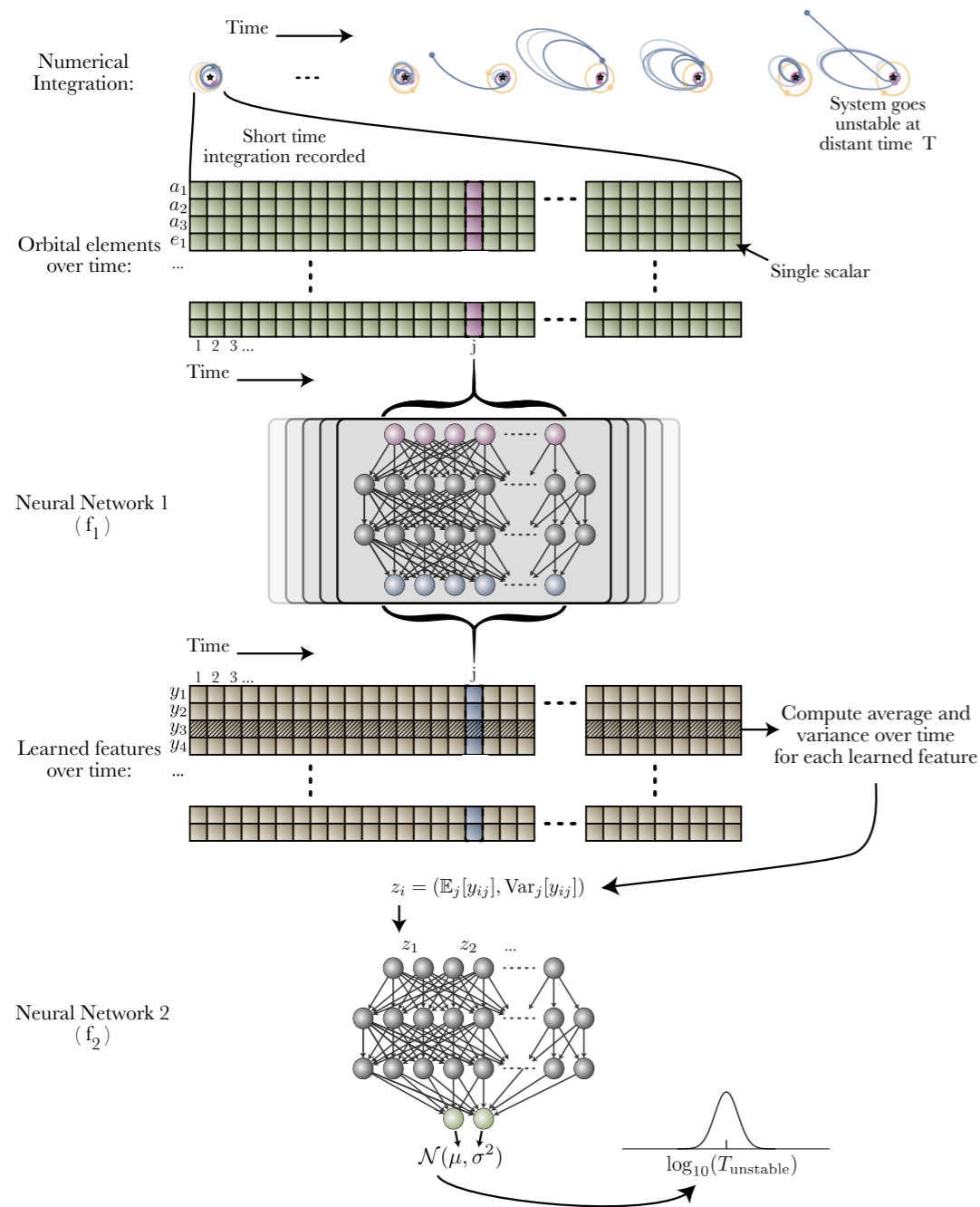


$$z_i = (\mathbb{E}_j[y_{ij}], \text{Var}_j[y_{ij}])$$

# Step 3: Predicting time of instability



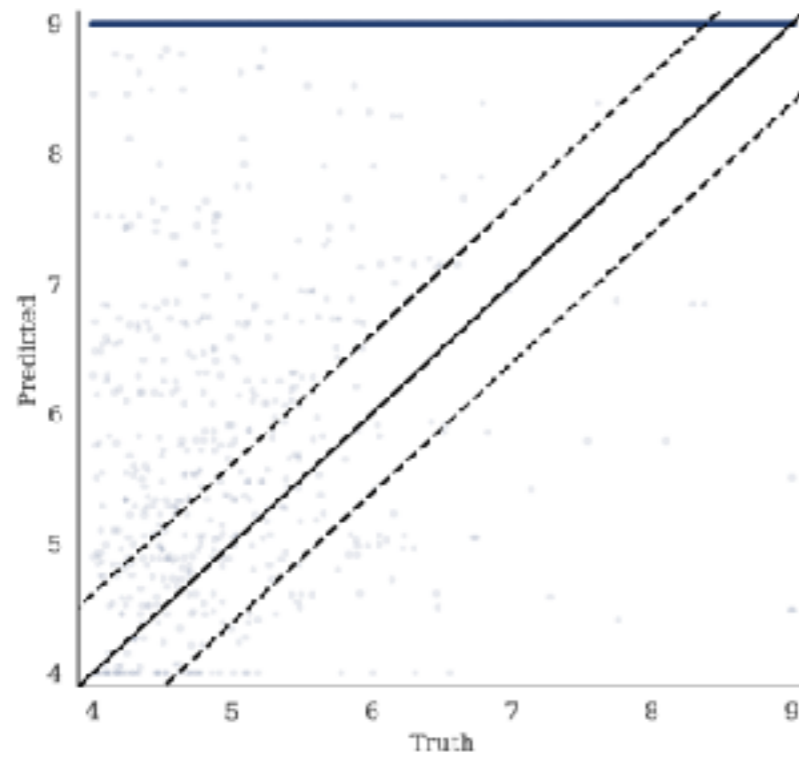
# Bayesian Neural Network



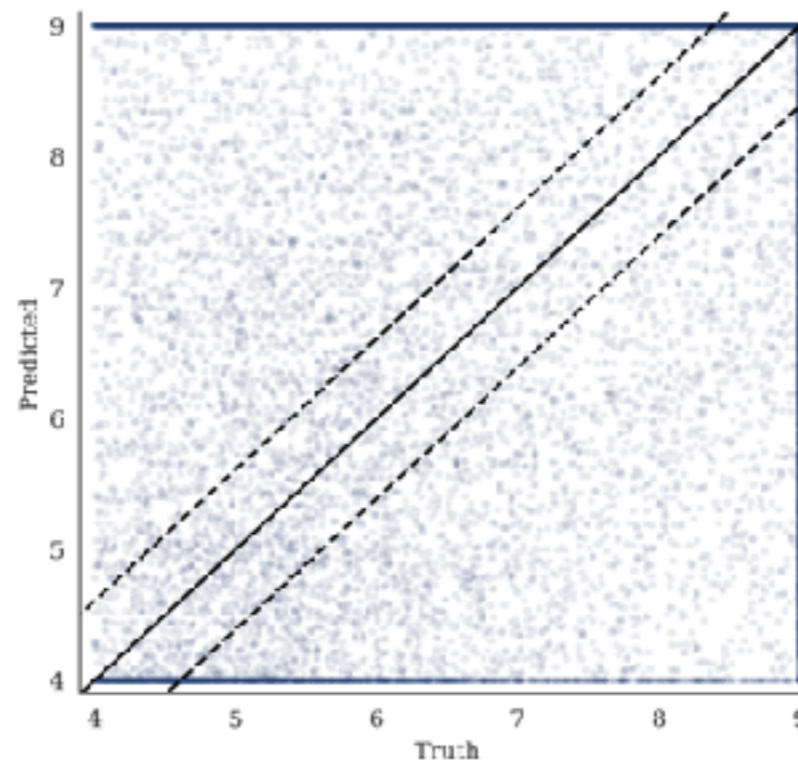
- ▶ Large training dataset: 113,543 simulation
- ▶ Generating training dataset is tricky
- ▶ Sampling a highly complex parameter space
- ▶ Computationally very expensive, billion orbits



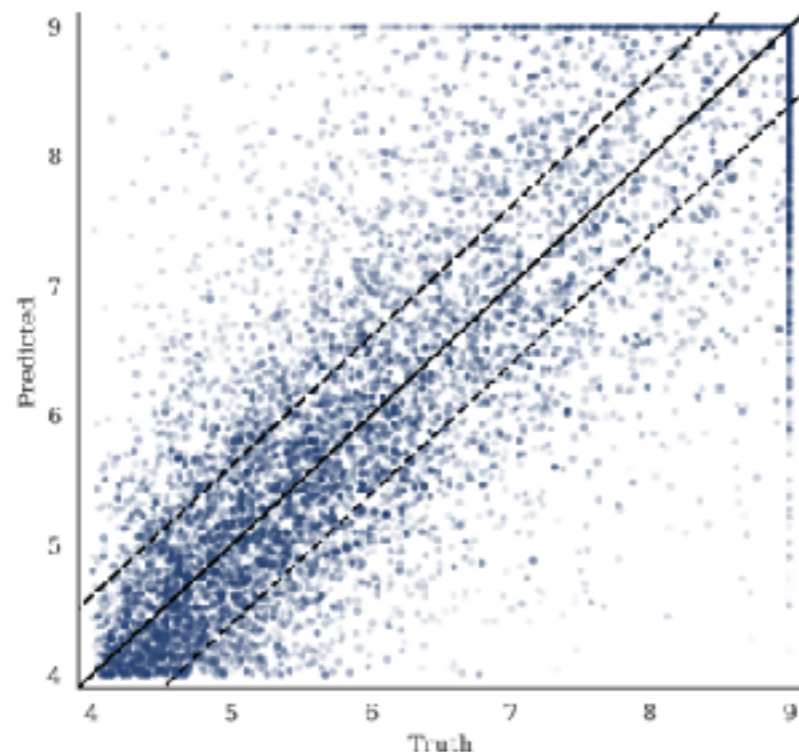
# Results



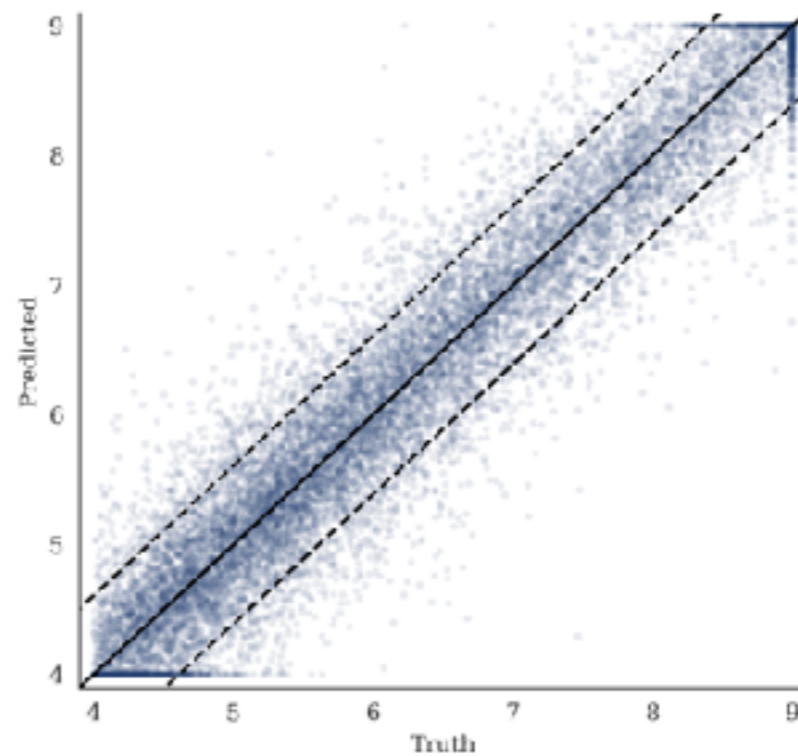
Petit et al. (2020)



Obertas et al. (2017)

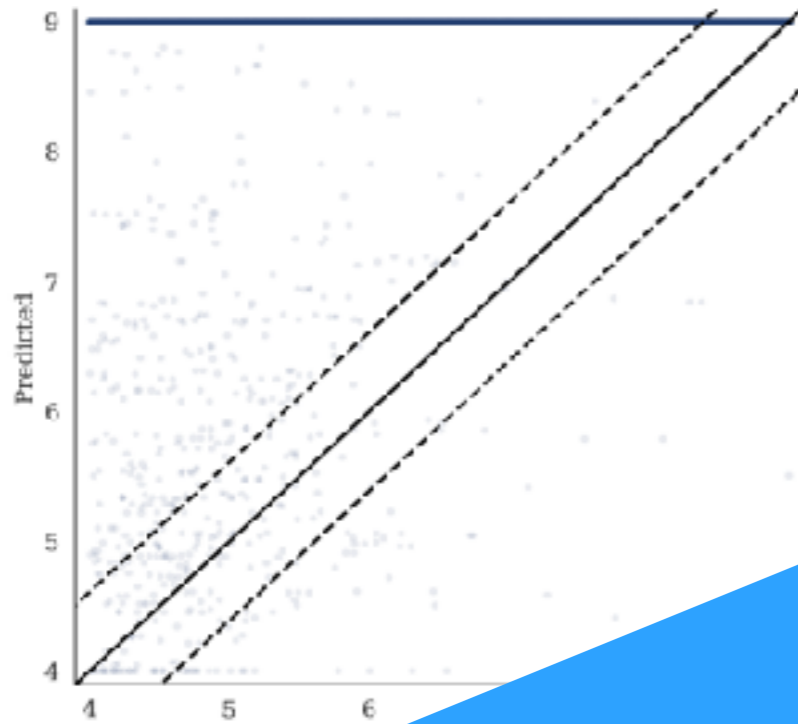


Our ML model, Cranmer et al (2021)

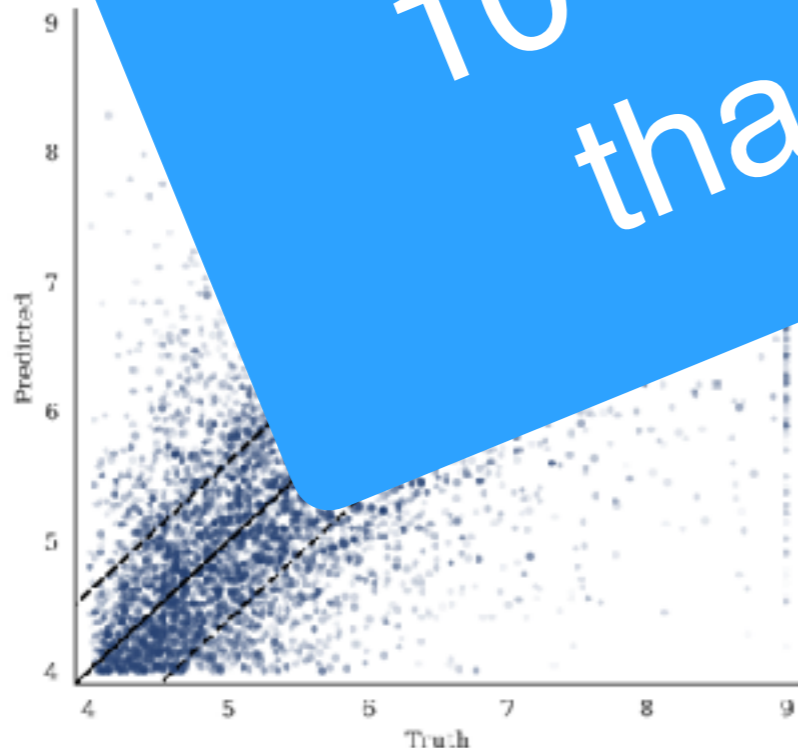


Theoretical limit

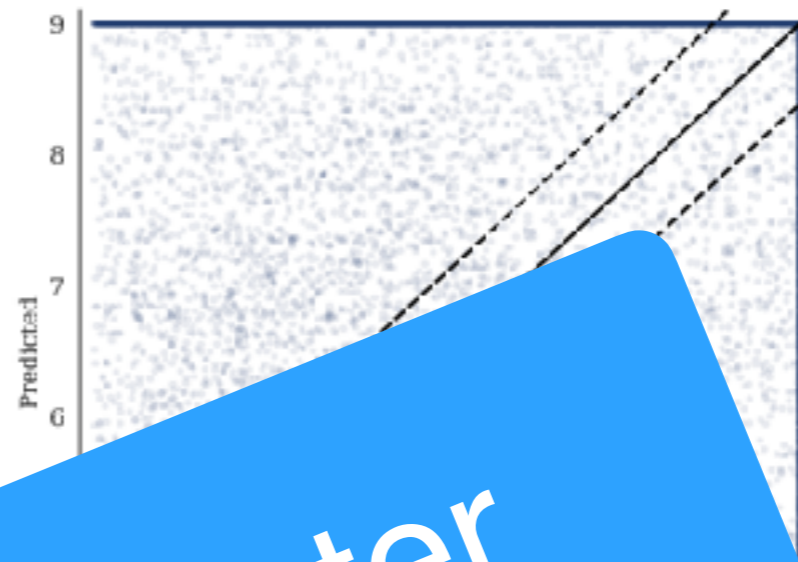
# Results



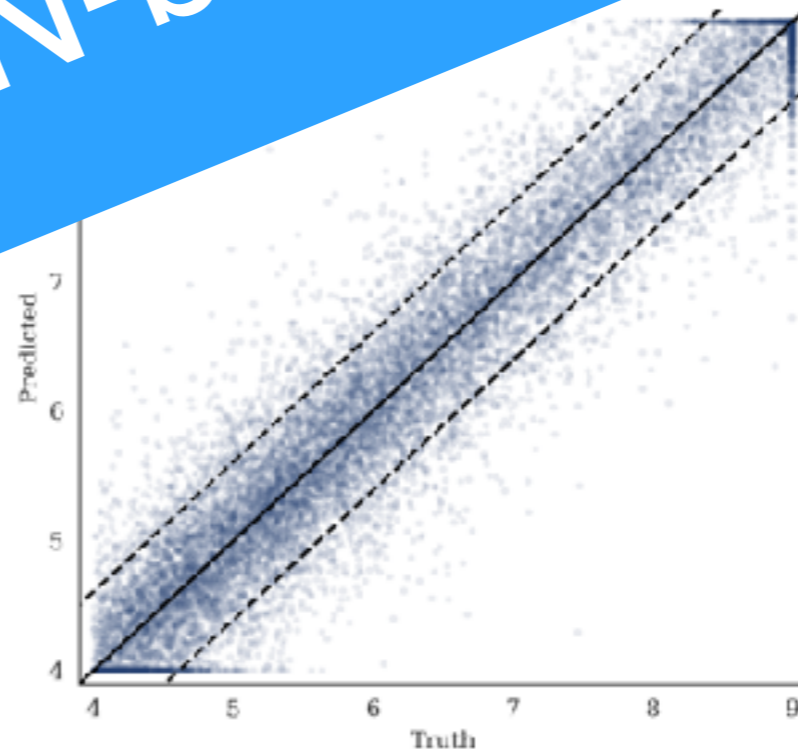
Petit



Our ML model, Cranmer et al (2021)



10<sup>5</sup> times faster  
than N-body

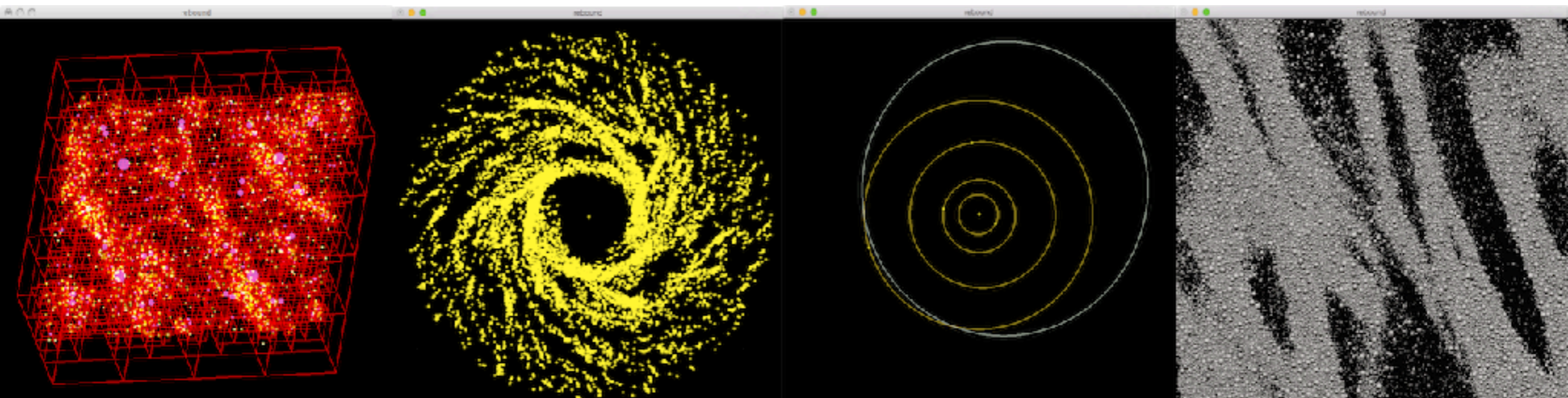


Theoretical limit

**REBOUND, REBOUNDx, ASSIST**

# REBOUND

- ▶ N-body integrator package
- ▶ Many different built-in integrators
- ▶ Planetary systems
- ▶ Collisional simulations of planetary rings
- ▶ Written in C with an easy to use python interface
- ▶ No dependencies



# WebAssembly

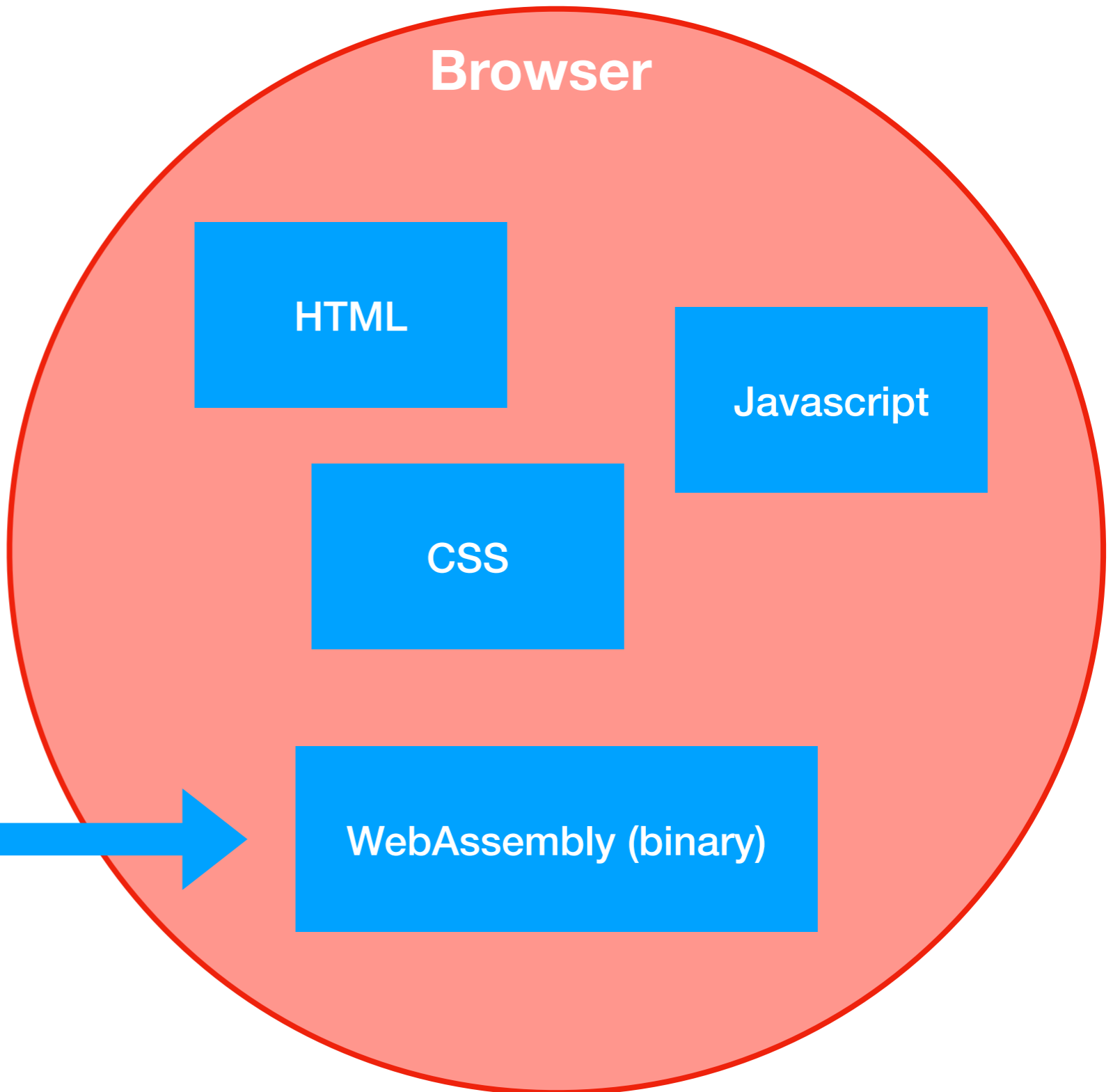
Source Code (C)



Compiler



WebAssembly (binary)

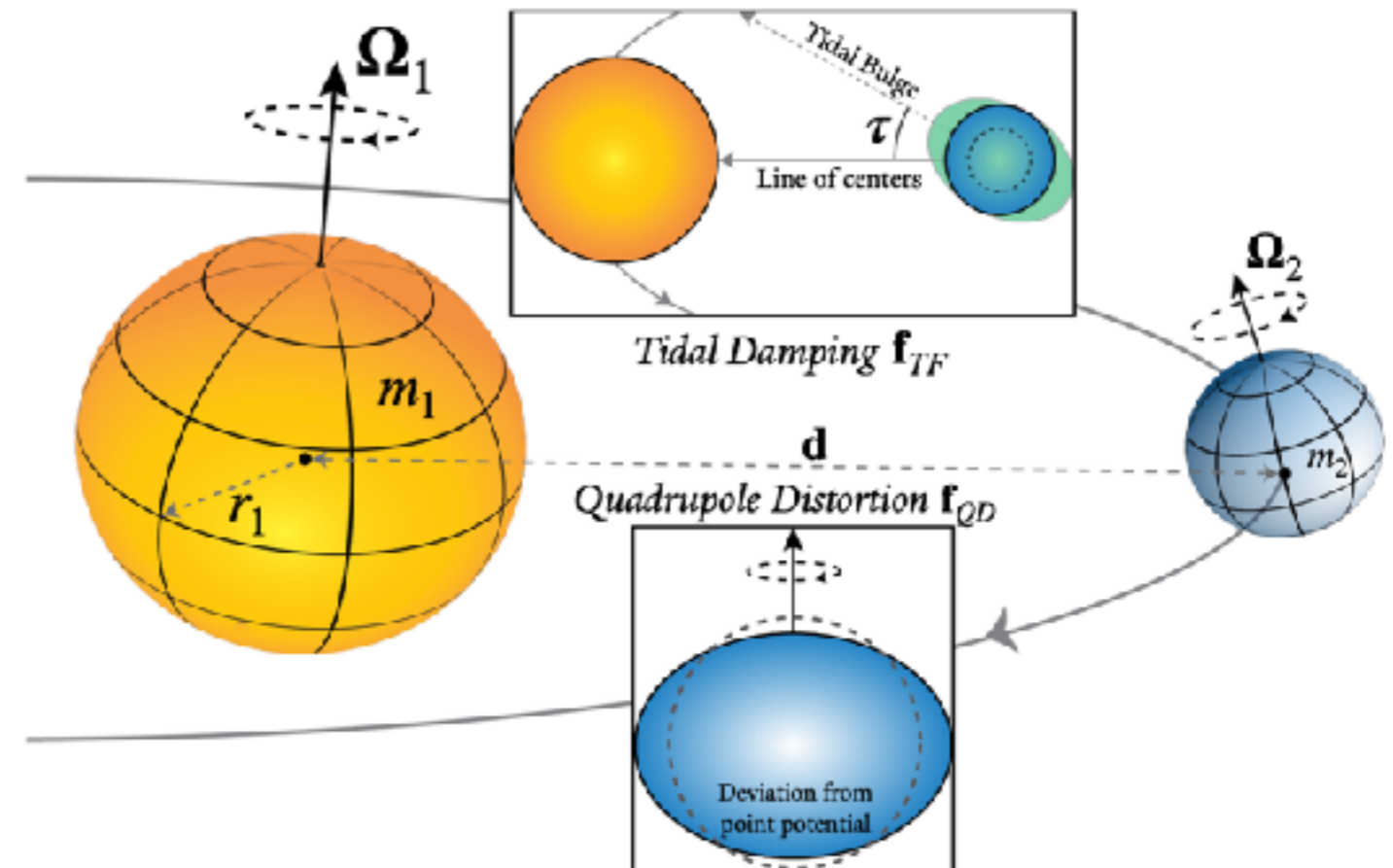


# REBOUNDx Add-on

- ▶ Development led by Dan Tamayo (Harvey Mudd)
- ▶ Incorporate additional physics into N-body simulations:
  - ▶ Orbit modifications/migration
  - ▶ General relativity
  - ▶ Radiation pressure, Yarkovsky effect
  - ▶ Gravitational harmonics
  - ▶ Tides
- ▶ Very easy to use
- ▶ Does a lot of smart things behind the scenes!

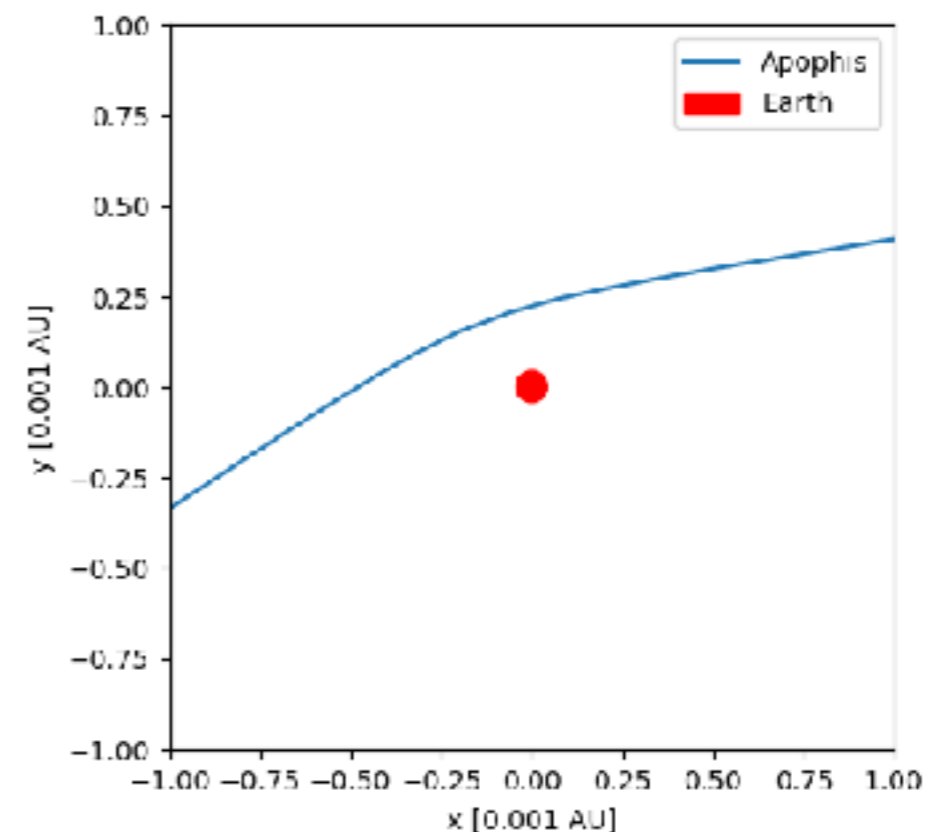
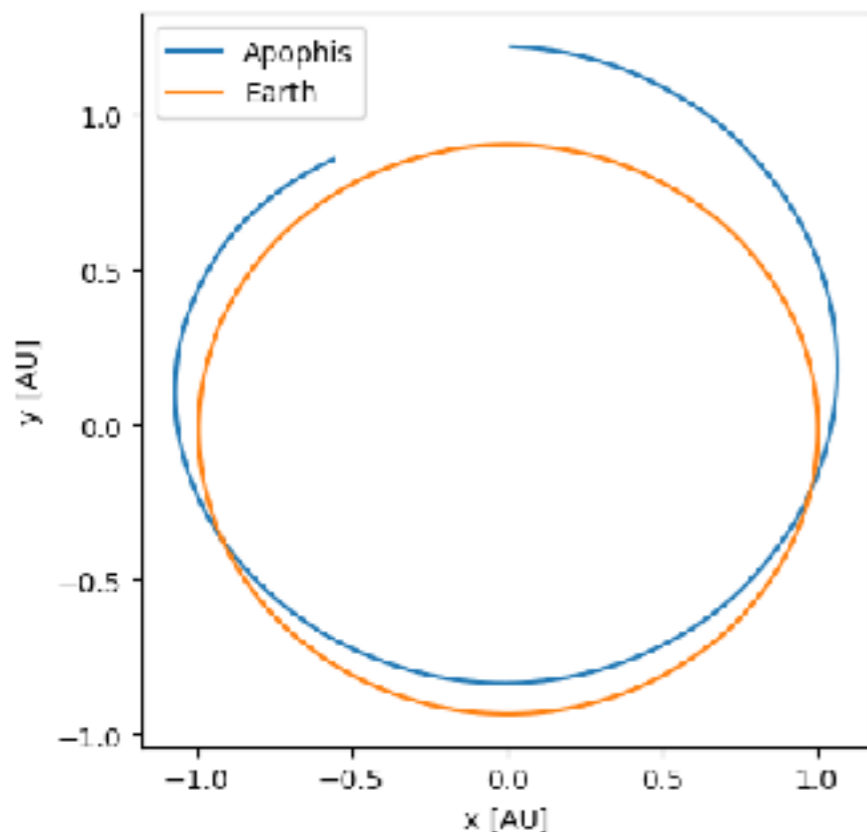
# REBOUNDx Tides

- ▶ Self-consistent spin, tidal and dynamical equations of motion
- ▶ Constant time lag approximation
- ▶ Part of REBOUNDx



# ASSIST Add-on

- ▶ Development led by Matt Holman (Minor Planet Center)
- ▶ Integration of asteroids, spacecrafts, artificial satellites in gravitational field of sun + planets (DE440 ephemeris)
- ▶ GR, radiation forces, higher order harmonics
- ▶ **Very** high accuracy, ~cm





# Conclusions

**Determining the stability of planetary systems is a very old problem. Analytic solutions cannot answer all questions.**

**Chaos leads to collisional trajectories in the Solar System.**

**Embedded Operator Splitting methods (EOS) are very easy to implement. Can be configured to be equivalent to: leap-frog, Wisdom-Holman, Mercury, SYMBA, and many new methods.**

**Our machine-learning classifier can predict the stability of planetary systems  $10^5$  times faster.**

**Use the REBOUND/REBOUNDx/ASSIST ecosystem for all your small N dynamics needs.**

# Light pollution and mega constellations



Plaskett Telescope

Aaron Boley,  
UBC



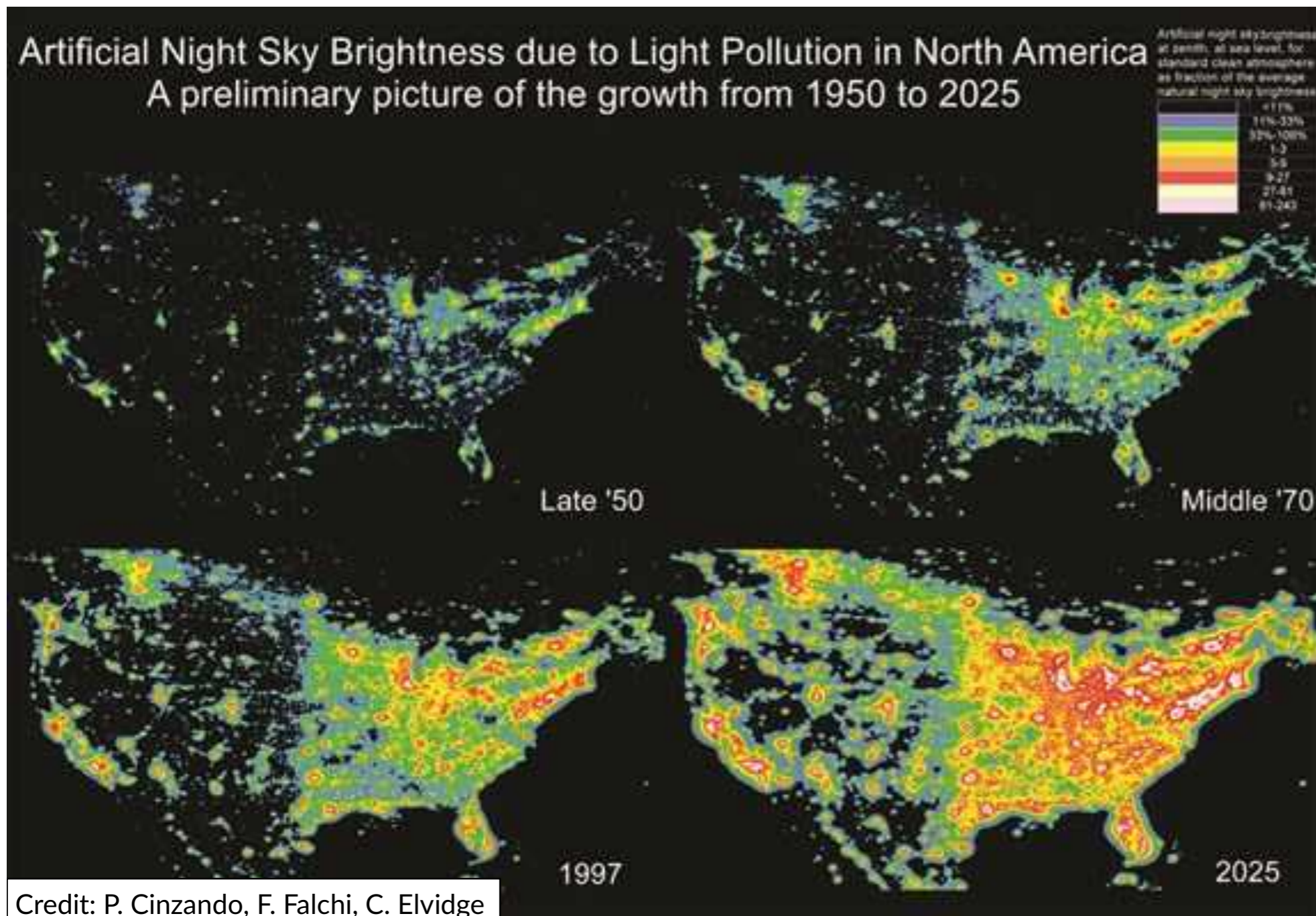
Samantha Lawler,  
U. of Regina



Hanno Rein,  
U. of Toronto



# Access to the night sky is restricted by urban light pollution



But there are many groups fighting!  
(ex: International Dark Sky Association)

LEDs (a sudden leap in technology access) took these groups by surprise

LEDs are good – use less energy for more light. BUT are massively over-used because they're cheap.

## A new source of globally visible light pollution



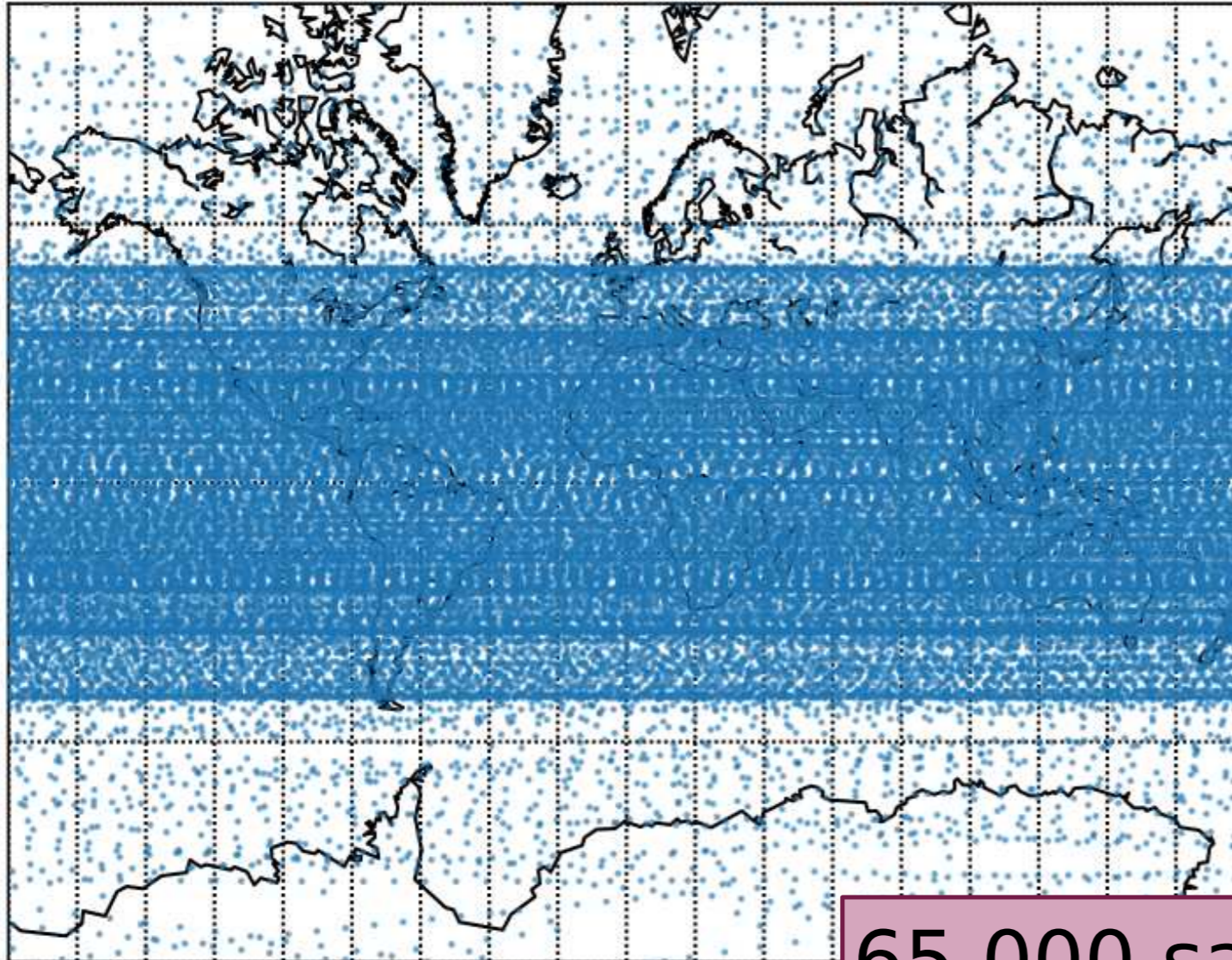
American private company SpaceX is launching batches of 60 satellites into low-Earth orbit **every 2-3 weeks.**

There are currently 3,633 Starlink satellites in orbit (out of 3,930 so far launched)

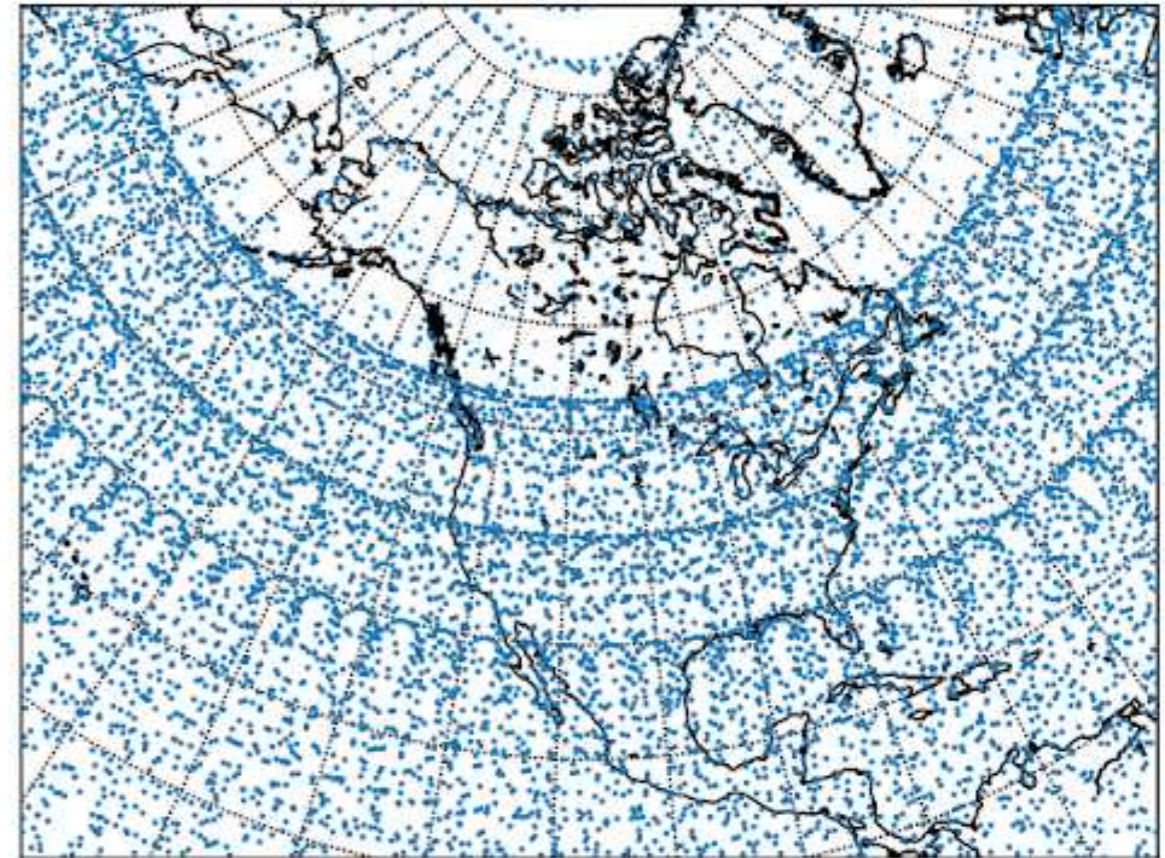
[Numbers from Jonathan McDowell's Starlink Statistics Page  
up to date as of 16 Feb 2023]

# How bad could it get?

Satellite Distribution (Lat-Lon Projection)

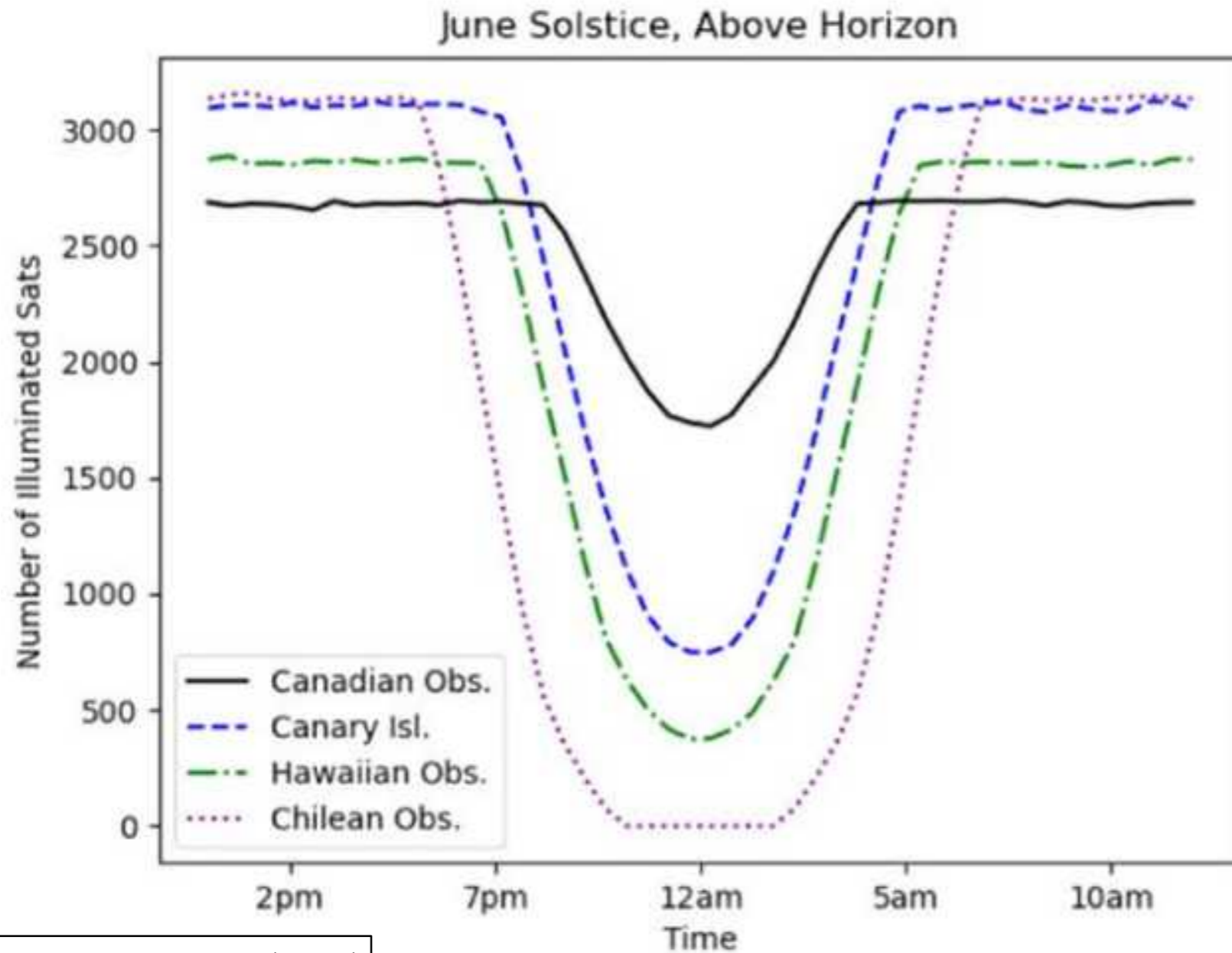


Satellite Distribution (Lat-Lon Projection)



65,000 satellites

# How bad could it get?



Lawler, Boley & Rein (2022)

Straightforward to calculate how many satellites are above the horizon and illuminated by sunlight.

... But how bright the satellites are when illuminated in orbit depends entirely on unknown engineering.

**Megaconstellations**  
Available for free on the AppStore