UNIVERSITY OF TORONTO

CENTRE FOR PLANETARY SCIENCES
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Recent advances in algorithms for long term integrations of planetary systems

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Talk Outline

The History of the N-body problem

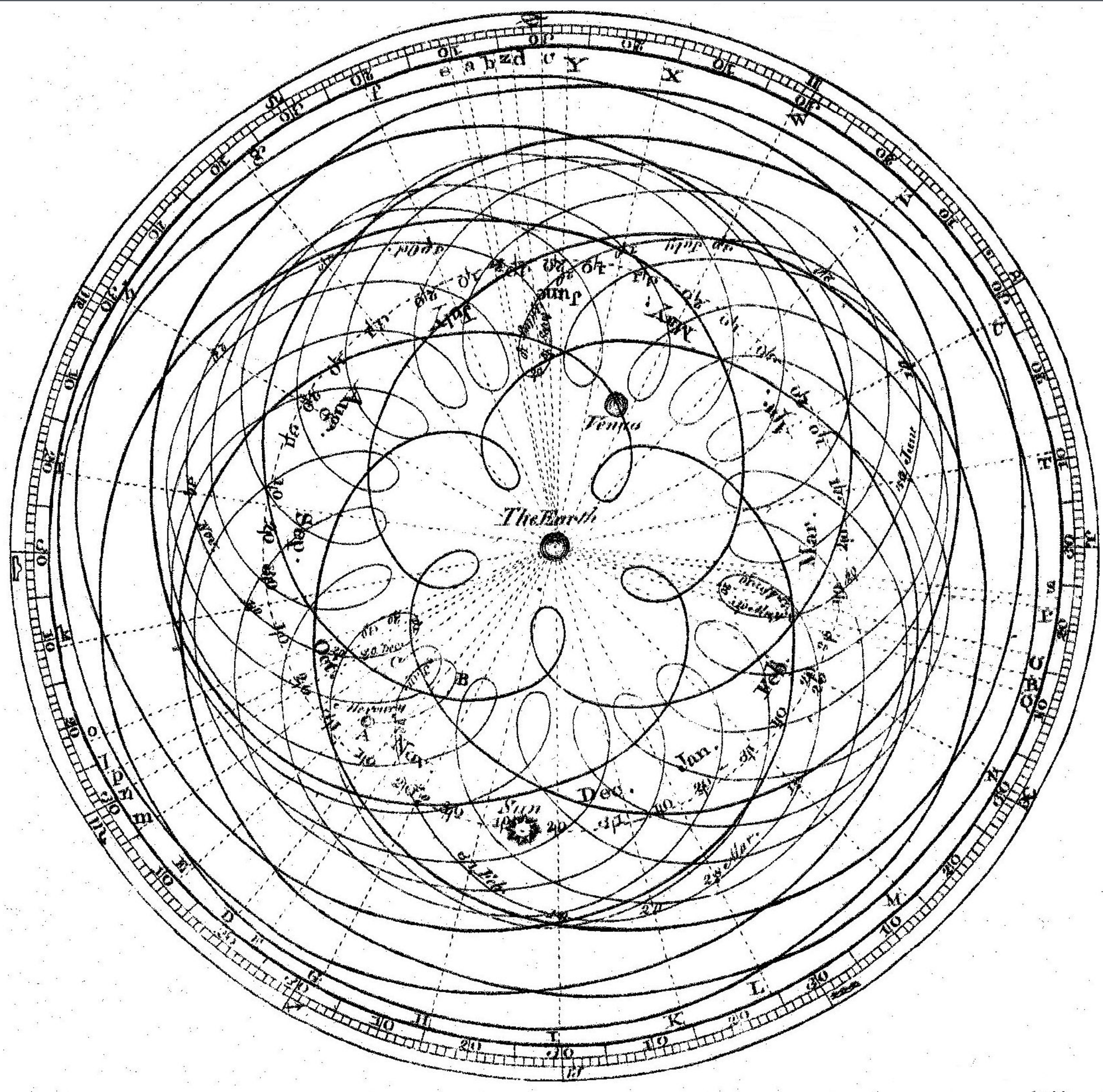
Operator splitting methods

What matters for the long-term evolution

The History of the N-body problem

Laskar (*Lagrange et la stabilité du Systéme solaire*, 2006), Laskar (2013)

Epicycles



Newton (1687)

$$\ddot{\mathbf{r}}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^3}$$

Newton (Opticks 1717,1730)

And to show that I do not take Gravity for an essential Property of Bodies, I have added one Question concerning its Cause, chosing to propose it by way of a Question, because I am not yet satisfied about it for want of Experiments. […]

For while comets move in very excentrick orbs in all manner of positions, blind fate could never make all the planets move one and the same way in orbs concentrick, some inconsiderable irregularities excepted, which may have risen from the mutual actions of comets and planets upon one another, and which will be apt to increase, till this system wants a reformation.

Evidence for irregularity/instability

On March 1st, 228 BC, at 4:23 am, mean Paris time, Saturn was observed two fingers under Gamma in Virgo.

+

Observations from 1590 and 1650.

=

Six million years ago Jupiter and Saturn were at the same distance from the Sun.

Explanations for the irregularities?

Euler was twice awarded a prize in 1748 and 1752 related to this problem by the Paris Academy of Sciences.

Lagrange thought that Euler’s calculations were wrong and did his own.

Laplace (1776)

Mr. Euler, in his second piece on the irregularities of Jupiter and Saturn, find it equal for both these planets. According to Mr. de Lagrange, on the contrary, the third volume of Mémoires de Turin, it is very different for these two bodies. […] I have some reasons to believe, however, that the formula is still not accurate. The one which I obtain is quite different. […] by substituting these values in the formula of the secular equation, I found absolutely zero, from which I conclude the alteration of the mean motion of Jupiter, if it exists, does not result from the action of Saturn.

Lagrange (in a letter to d'Alembert, 1775)

I am ready to give a complete theory for the variations of the elements of the planets under their mutual action. That Mr. de la Place did on this subject I liked, and I flatter myself that he will not be offended if I do not hold the kind of promise that I made to completely abandon this subject to him; I could not resist to the desire to look into it again, but I am no less charmed that he is also working on it on his side; I am even very eager to read his subsequent research on this topic, but I do ask him not to send me any manuscript and send them to me only in printed form; I would be obliged that you tell him, with a thousand compliments from my side.

Secular Dynamics, Lagrange (1774)

1) Averaging over short time scales.

2) Perturbation theory.

No semi-major axis changes to first and also second order (Poisson, Haretu and Poincaré) in the expansion.

This still contradicts Ptolemy's observations from antiquity.

	Lagrange	**Brown & Rein (in prep)**
s_1	5.98	5.59
s_2	6.31	7.05
s_3	19.80	18.84
s_4	18.31	17.74
s_5	0	0
s_6	25.34	26.35
s_7	-	2.99
s_8	-	0.69

Laplace (1785)

Simple energy argument implies:

$$\frac{m_J}{a_J} + \frac{m_S}{a_S} = const$$

Thus, can be confident that the change in orbits must be due to mutual interactions.

He’s also shown, no secular terms. Hence must be short period.

Near 5:2 mean motion resonance. Period of 900 years.

Why is 900 years important?

Stability

Poincaré (1897)

Those who are interested in the progress of celestial mechanics, […] must feel some astonishment at seeing how many times the stability of the Solar System has been demonstrated.
Lagrange established it first, Poisson has demonstrated it again, other demonstrations came afterwards, others will come again. Were the old demonstrations insufficient, or are the new ones unnecessary?
The astonishment of those people would probably double, if they would be told that perhaps one day a mathematician will demonstrate, by a rigorous reasoning, that the planetary system is unstable.

Le Verrier (1840, 1841)

Follows up on the work of Lagrange and Laplace

Goes to higher order.

Discovered small divisor problem: third order could be larger than second order terms

Poincaré

The terms of these series, in fact, decrease first very quickly and then begin to grow, but as the Astronomers's stop after the first terms of the series, and well before these terms have stop to decrease, the approximation is sufficient for the practical use. The divergence of these expansions would have some disadvantages only if one wanted to use them to rigorously establish some specific results, as the stability of the Solar System.

Kolmogorov (1954), Arnold (1963), Moser (1962)

Kolmogorov showed that convergent perturbation series exists

Skipping many subtleties (degeneracy, small masses, slow Arnold diffusion)

In short: not useful for determining the stability of our Solar System with a very specific set of initial conditions

Operator splitting methods
Fundamentals

Rein, Brown & Tamayo (2019), Rein, Tamayo & Brown (2019)

Fundamentals

Poisson Bracket of two functions f, g of the canonical coordinates q, p.

$$\{g, h\} = \sum_i \left(\frac{\partial g}{\partial q_i} \frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q_i} \right)$$

Allows us to write Hamilton's equations as

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\} \qquad \text{and} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}$$

Allows us to write the time derivate of *any* function of q and p as

$$\dot{g} = \{g, H\}$$

Fundamentals

Introduce

$$y(t) = (q_1(t), \ldots, q_N(t), p_1(t), \ldots, p_N(t))$$

The differential equation for the N-body problem is then

$$\dot{y}(t) = \{y, H\}$$

A bit more notations. Lie derivative:

$$\mathcal{L}_H = \{\cdot, H\}$$

Define a solution operator:

$$\varphi_t^{[H]}(y_0)$$

Notation roughly follows Hairer, Lubich & Wanner 2006

Fundamentals

The differential equation we're trying to solve

$$\dot{y}(t) = \mathcal{L}_H \, y(t)$$

A formal way to write down the solution operator

$$\begin{aligned}\varphi_t^{[H]}(y_0) &= \exp\left(t\mathcal{L}_H\right) \mathrm{Id}(y_0) \\ &= \left(\mathrm{Id} + t\mathcal{L}_H \mathrm{Id} + \frac{1}{2} t^2 \mathcal{L}_H \mathcal{L}_H \mathrm{Id} + \ldots\right)(y_0)\end{aligned}$$

Only a formal solution!

Notation roughly follows Hairer, Lubich & Wanner 2006

Fundamentals

Splitting method

$$\dot{y} = \mathcal{L}_H(y)$$

$$H = A + B$$

$$\dot{y} = (\mathcal{L}_A + \mathcal{L}_B)(y) = \mathcal{L}_A(y) + \mathcal{L}_B(y).$$

We can now consider two differential equations

$$\dot{y} = \mathcal{L}_A(y) \quad \text{and} \quad \dot{y} = \mathcal{L}_B(y).$$

Example!

$$H = \frac{1}{2}p^2 + U(q) \qquad\qquad y(t) = (q(t), p(t))$$

Let's split the Hamiltonian:

$$A = \frac{1}{2}p^2 \qquad B = U(q)$$

$$\dot{y} = \mathcal{L}_A\, y \qquad \dot{y} = \mathcal{L}_B\, y$$

$$\dot{y} = \{y, A\} \qquad \dot{y} = \{y, B\}$$

$$\dot{q} = p \qquad \dot{q} = 0$$

$$\dot{p} = 0 \qquad \dot{p} = -\nabla U(q)$$

Solutions are trivial!

Example!

$$\dot{q} = p \qquad \varphi_t^{[A]}(y_0) = (q_0 + t \cdot p_0, p_0)$$
$$\dot{p} = 0$$

$$\dot{q} = 0 \qquad \varphi_t^{[B]}(y_0) = (q_0, p_0 - t \cdot \nabla U(q))$$
$$\dot{p} = -\nabla U(q)$$

Idea of a splitting scheme is:

$$\varphi_t^{[H]}(y_0) \approx \varphi_t^{[B]} \varphi_t^{[A]}(y_0)$$

Example!

$$\varphi_t^{[A]}(y_0) = (q_0 + t \cdot p_0, p_0)$$

$$\varphi_t^{[B]}(y_0) = (q_0, p_0 - t \cdot \nabla U(q))$$

$$\exp\left(t\mathcal{L}_A\right)\exp\left(t\mathcal{L}_B\right) = \exp\left(t\mathcal{L}_A + t\mathcal{L}_B + \frac{1}{2}t^2\left[\mathcal{L}_A, \mathcal{L}_B\right] + \mathcal{O}(t^3)\right)$$

BCH

Can show that

$$[\mathcal{L}_A, \mathcal{L}_B] = \mathcal{L}_{\{A,B\}}$$

So, we're solving the EoM for the following Hamiltonian *exactly*

$$H' = A + B + \frac{1}{2}t\{A, B\} + \mathcal{O}(t^2)$$

Example!

We're solving the EoM for the following Hamiltonian *exactly*

$$H' = A + B + \frac{1}{2}t\{A, B\} + \mathcal{O}(t^2)$$

for our specific example:

$$\begin{aligned} H' &= \frac{1}{2}p^2 + U(q) + \frac{1}{2}t\left\{\frac{1}{2}p^2, U(q)\right\} + \mathcal{O}(t^2) \\ &= H - \frac{t}{2}p\nabla U(q) + \mathcal{O}(t^2) \end{aligned}$$

Works for small t, thus

$$t \to dt$$

and repeat many times.

Operator splitting methods for the N-body problem

Rein, Brown & Tamayo (2019), Rein, Tamayo & Brown (2019)

Wisdom-Holman integrator

Splitting the Hamiltonian:

$$H = A + \epsilon B$$

Wisdom-Holman integrator:

$$\mathrm{WH} = e^{\frac{1}{2}A} e^{B} e^{\frac{1}{2}A}.$$

Hamiltonian we're solving is:

$$\begin{aligned} H' =& A + \epsilon B - \epsilon \frac{dt^2}{24} \{A, \{A, B\}\} - \epsilon^2 \frac{dt^2}{12} \{B, \{A, B\}\} \\ & + \mathcal{O}(\epsilon dt^4) + \mathcal{O}(\epsilon^2 dt^4) + \ldots \end{aligned}$$

Wisdom-Holman integrator

Wisdom-Holman integrator:

$$\mathrm{WH} = e^{\frac{1}{2}A} e^{B} e^{\frac{1}{2}A}.$$

Extension to higher order

$$\begin{aligned}\mathrm{SABA}(10{,}6{,}4) = {} & e^{a_1 A} e^{b_1 B} e^{a_2 A} e^{b_2 B} e^{a_3 A} e^{b_3 B} e^{a_4 A} e^{b_4 B} e^{a_5 A} \\ & e^{b_5 B} e^{a_6 A} e^{b_6 B} e^{a_7 A} e^{b_7 B} e^{a_8 A} e^{b_8 B} e^{a_9 A}\end{aligned}$$

Which solves

$$H' = A + \epsilon B + \mathcal{O}(\epsilon dt^{10}) + \mathcal{O}(\epsilon^2 dt^6) + \mathcal{O}(\epsilon^3 dt^4) + \ldots$$

McLachlan (1995), Laskar & Robutel (2001), Blanes et al. (2015)

Survey (Rein et al. 2019a)

Name and synonyms	Main references	Start up/shut down	One time-step	Cost	Only A, B	$\mathcal{O}\left(\epsilon\, \mathrm{d}t^{?}\right)$	$\mathcal{O}\left(\epsilon^2\, \mathrm{d}t^{?}\right)$	$\mathcal{O}\left(\epsilon^3\, \mathrm{d}t^{?}\right)$	Implemented in REBOUND
WH ★ SABA1 (d) WHFAST (e) M2 (b)	(a), (e), (f)	–	$A\,B\,A$	1	✓	2			✓
WHCp CM2 (b)	(b), (c), (f)	$C^{(1)}_{[p]}$	$A\,B\,A$	1	✓	$p+1$	2		✓(up to $p=17$)
WHCCKI (ideal kernel)	(b)	$C^{(2)}_{*}\,C^{(1)}_{*}$	$A\,B_{*}\,A$ $* =$ not possible			∞	∞	4	(not possible)
WHCKC (comp. kernel)	(b)	$C^{(1)}_{[17]}$	$A\,(B\,A)^5$	5	✓	18	4		✓
WHCKM (mod. kick kernel) CMM4 (b)	(b)	$C^{(1)}_{[17]}$	$A\,\widehat{B}\,A$	1		18	4		✓
WHCKL ★ (lazy impl. kernel)	(b)	$C^{(1)}_{[17]}$	$A\,\widehat{B\,B}\,A$	2	✓	18	4	3	✓
WHCCKC (comp. kernel)	(b)	$C^{(2)}\,C^{(1)}_{[17]}$	$A\,(B\,A)^5$	5	✓	18	4		✓
WHCCKM (mod. kick kernel) WHCK (f)	(b) (f)	$C^{(2)}\,C^{(1)}_{[17]}$	$A\,\widehat{B}\,A$	1		18	4		✓
WHCCKL (lazy impl. kernel)	(b)	$C^{(2)}\,C^{(1)}_{[17]}$	$A\,\widehat{B\,B}\,A$	2	✓	18	4	3	✓
SABA2	(d)	–	$A\,B\,A\,B\,A$	2	✓	4	2		✓
SABA3	(d)	–	$A\,(B\,A)^3$	3	✓	6	2		✓
SABA4	(d)	–	$A\,(B\,A)^4$	4	✓	6	2		✓
SABAC2	(d)	–	$\widehat{B}\,A\,(B\,A)^2\,\widehat{B}$	3		4	4		✓
SABAC3	(d)	–	$\widehat{B}\,A\,(B\,A)^3\,\widehat{B}$	4		6	4		✓
SABAC4	(d), (g)	–	$\widehat{B}\,A\,(B\,A)^4\,\widehat{B}$	5		6	4		✓
SABACL4 ★ (lazy impl. cor.)	This paper	–	$\widehat{B\,B}\,A\,(B\,A)^4\,\widehat{B\,B}$	6		6	4	3	✓
SABA(10,4)	(h)	–	$A\,(B\,A)^7$	7	✓	10	4		✓
SABA(8,6,4)	(h)	–	$A\,(B\,A)^7$	7	✓	8	6	4	✓
SABA(10,6,4) ★	(h)	–	$A\,(B\,A)^8$	8	✓	10	6	4	✓

Alternative 1
Dynamical Systems Approach

Wisdom 2018, Wisdom, Holman & Touma 1996

Wisdom-Holman integrator revisited

Splitting the Hamiltonian:

$$H = A + \epsilon B$$

The actual Hamiltonian solved

$$H' = A + \epsilon B - \epsilon \frac{dt^2}{24} \{A, \{A, B\}\} - \epsilon^2 \frac{dt^2}{12} \{B, \{A, B\}\} + \mathcal{O}(\epsilon dt^4) + \mathcal{O}(\epsilon^2 dt^4) + \ldots$$

Wisdom-Holman integrator revisited

Wisdom's "dynamical systems approach" looks at

$$
\begin{aligned}
H' &= A + \epsilon B + \epsilon B \sum_{n\neq 0} \cos(n\Omega t) \\
&= A + 2\pi\delta_{2\pi}(\Omega t)\epsilon B
\end{aligned}
$$

This Hamiltonian is now trivial to solve

$$
\xrightarrow{\quad e^A \quad} \overset{e^B}{\delta(\Omega t)} \xrightarrow{\quad e^A \quad} \overset{e^B}{\delta(\Omega t)} \xrightarrow{\quad e^A \quad} \overset{e^B}{\delta(\Omega t)} \xrightarrow{\qquad\quad}
$$

Alternative 2
Quadratures

Laskar & Robutel 2001

Quadratures

Trying to solve an ODE

$$\dot{y}(t) = \mathcal{L}_H y(t)$$

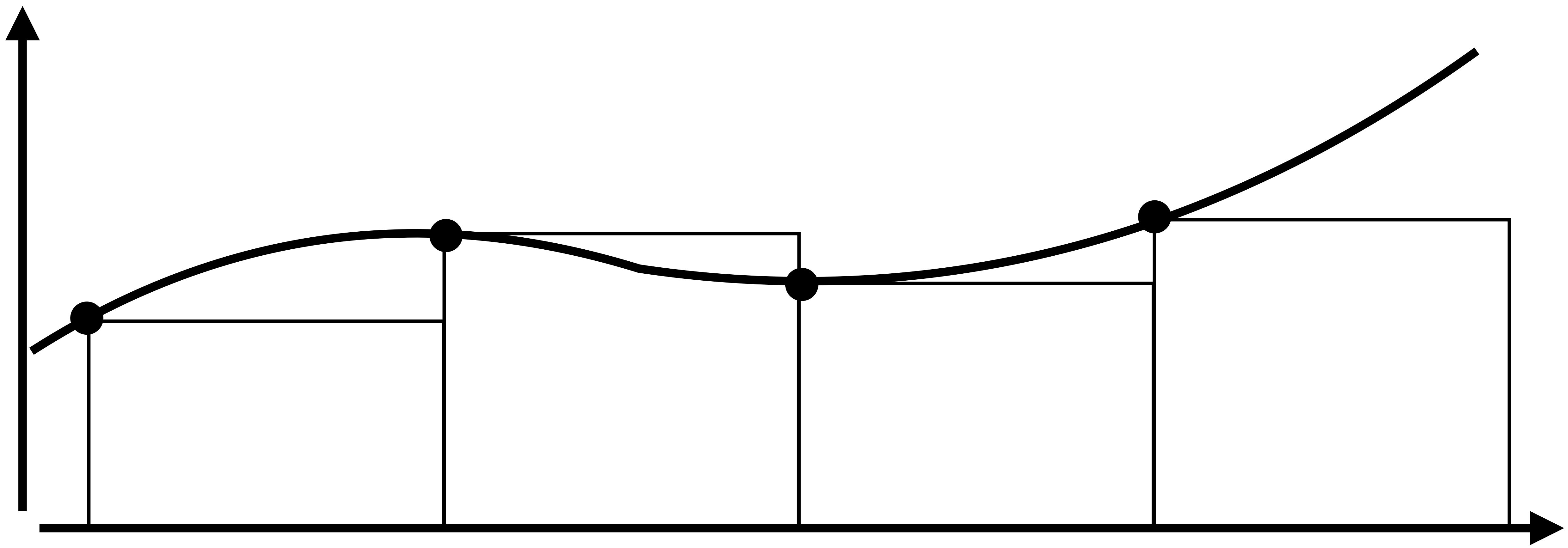
We can also integrate both sides to get

$$\int \dot{y}(t)dt = \int \mathcal{L}_H y(t)dt$$

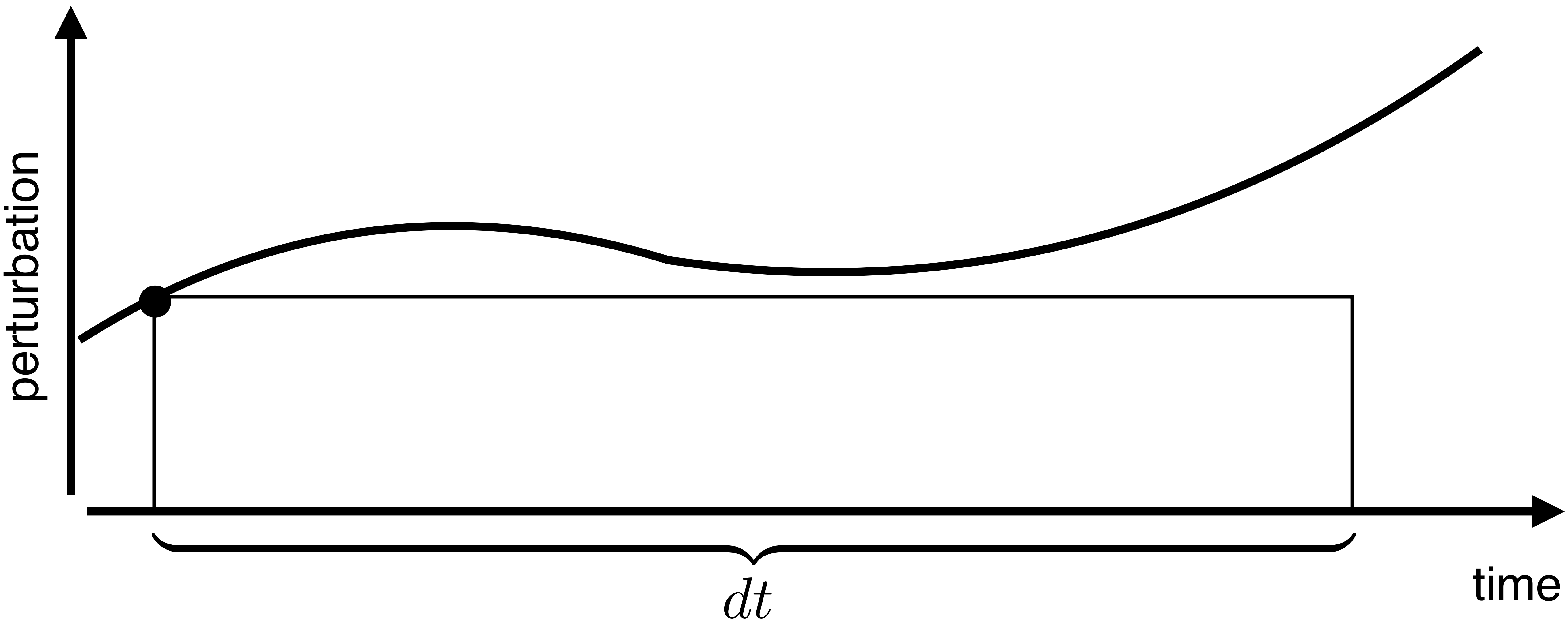
Thus

$$y(t) = \int \mathcal{L}_H y(t)dt$$

Quadratures



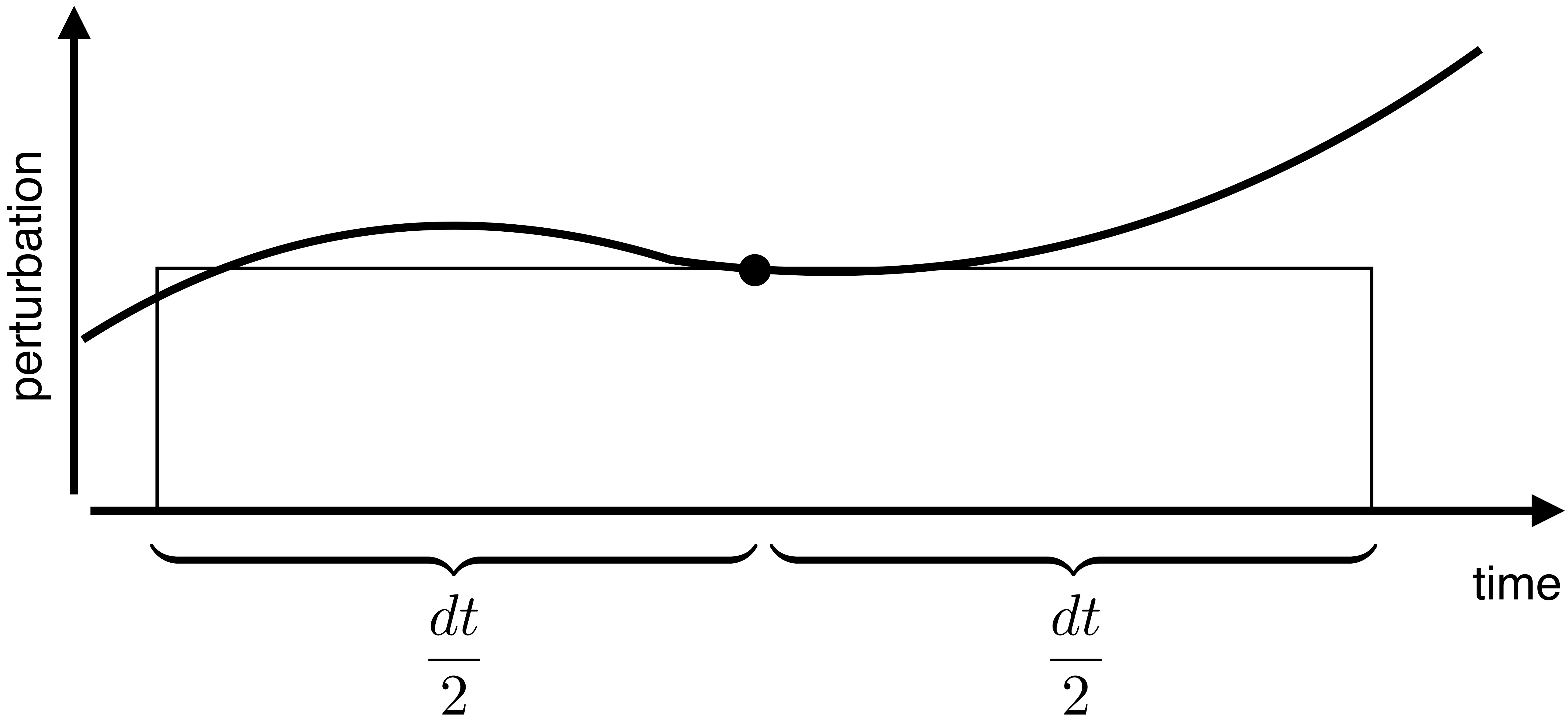
Quadratures



One-sided Riemann sum is equivalent to (symplectic) Euler method

$$\mathrm{Euler} = e^{A} e^{B}$$

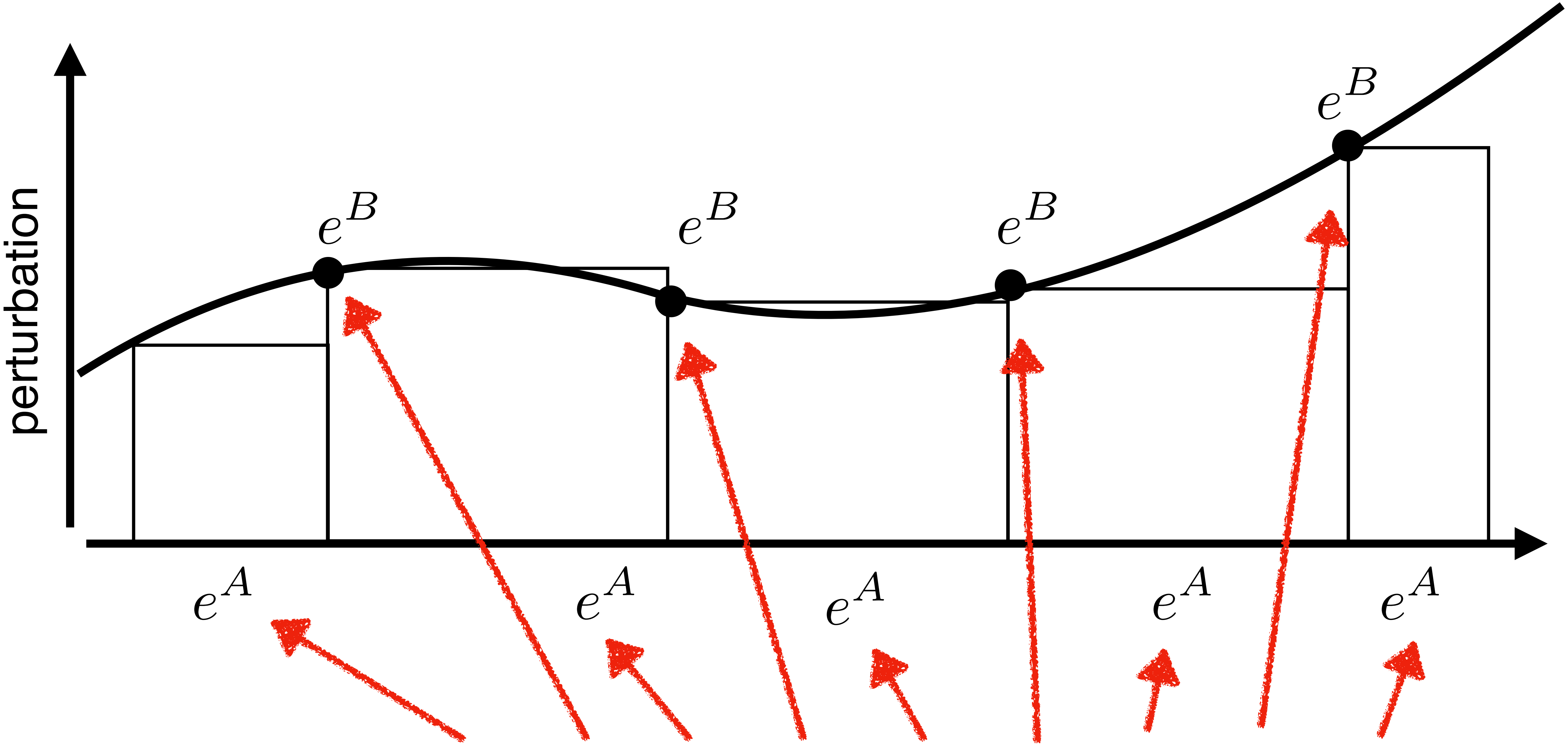
Quadratures



Mid-point method is equivalent to the WH method

$$\mathrm{WH} = e^{\frac{1}{2}A} e^{B} e^{\frac{1}{2}A}.$$

Quadratures



$$\mathrm{SABA4} = e^{a_1 A} e^{b_1 B} e^{a_2 A} e^{b_2 B} e^{a_3 A} e^{b_3 B} e^{a_4 A} e^{b_4 B} e^{a_5 A}$$

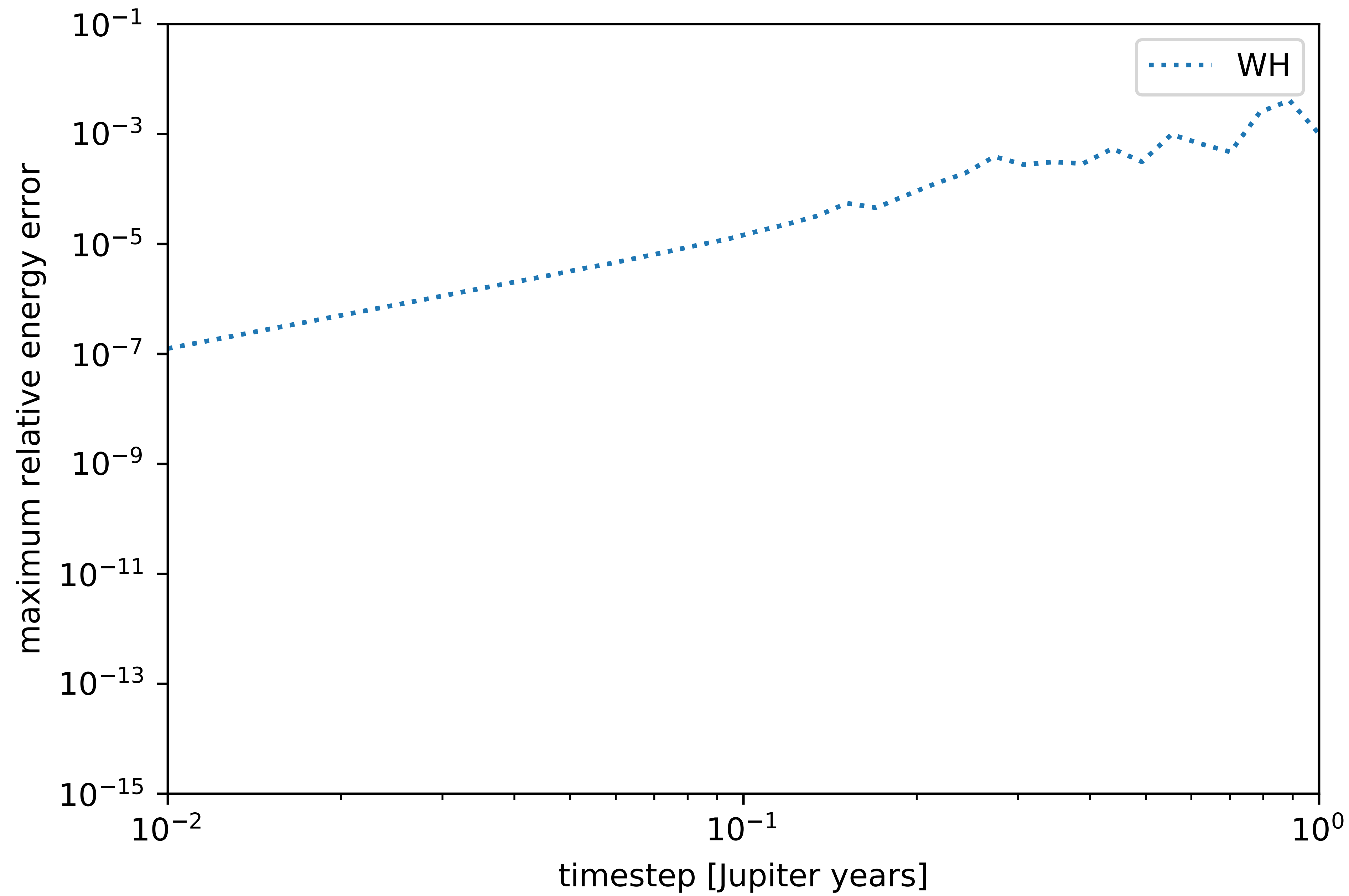
equivalent to a 4 point Gauß-Legendre quadrature

What matters for long-term integrations?

Rein, Brown & Tamayo (2019), Rein, Tamayo & Brown (2019)

Wisdom-Holman

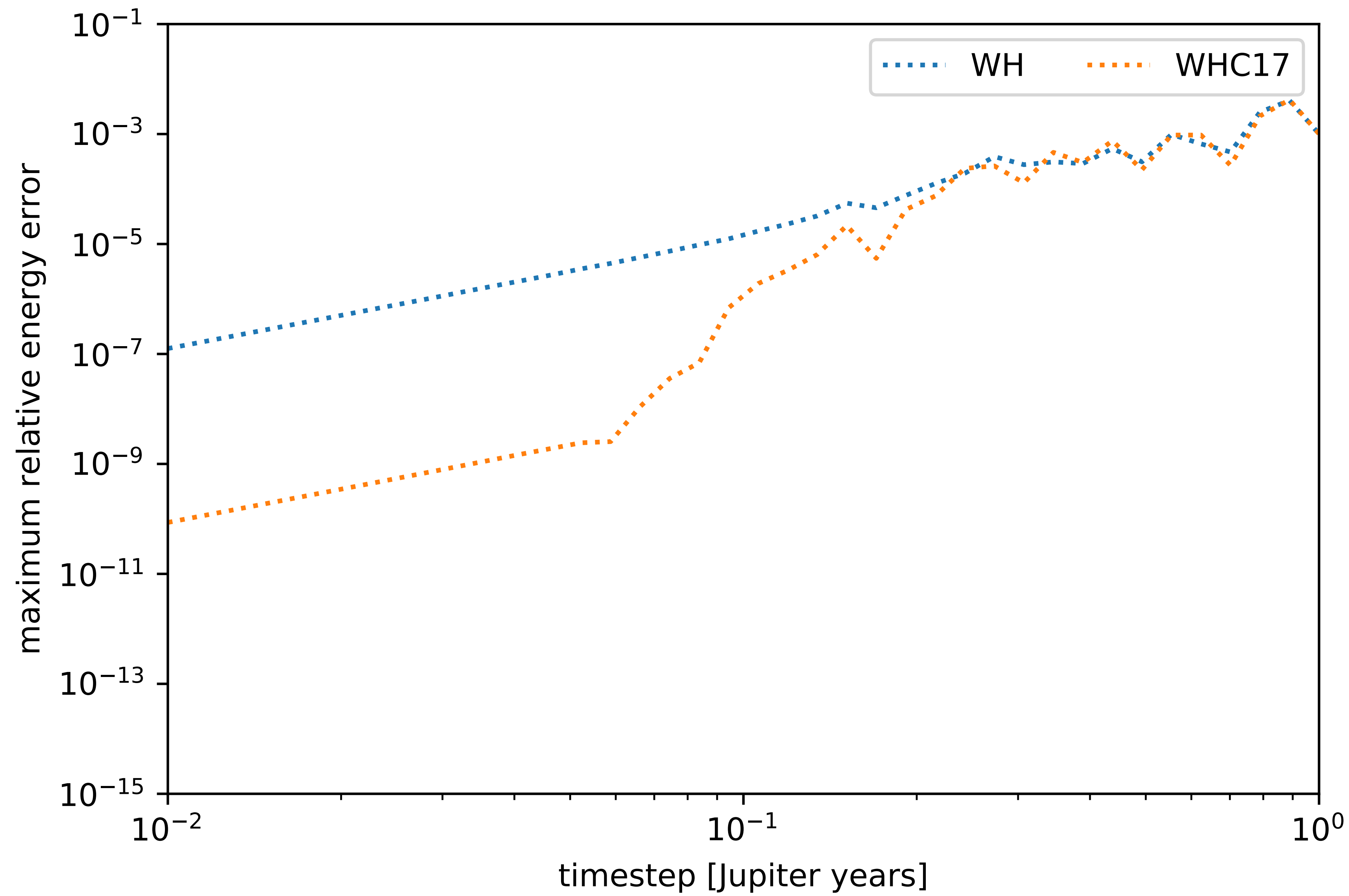
$$H' = A + \epsilon B - \epsilon \frac{dt^2}{24}\{A,\{A,B\}\} - \epsilon^2 \frac{dt^2}{12}\{B,\{A,B\}\} + \mathcal{O}(\epsilon dt^4) + \mathcal{O}(\epsilon^2 dt^4) + \ldots$$



Rein, Tamayo & Brown (2019)

Wisdom-Holman with correctors

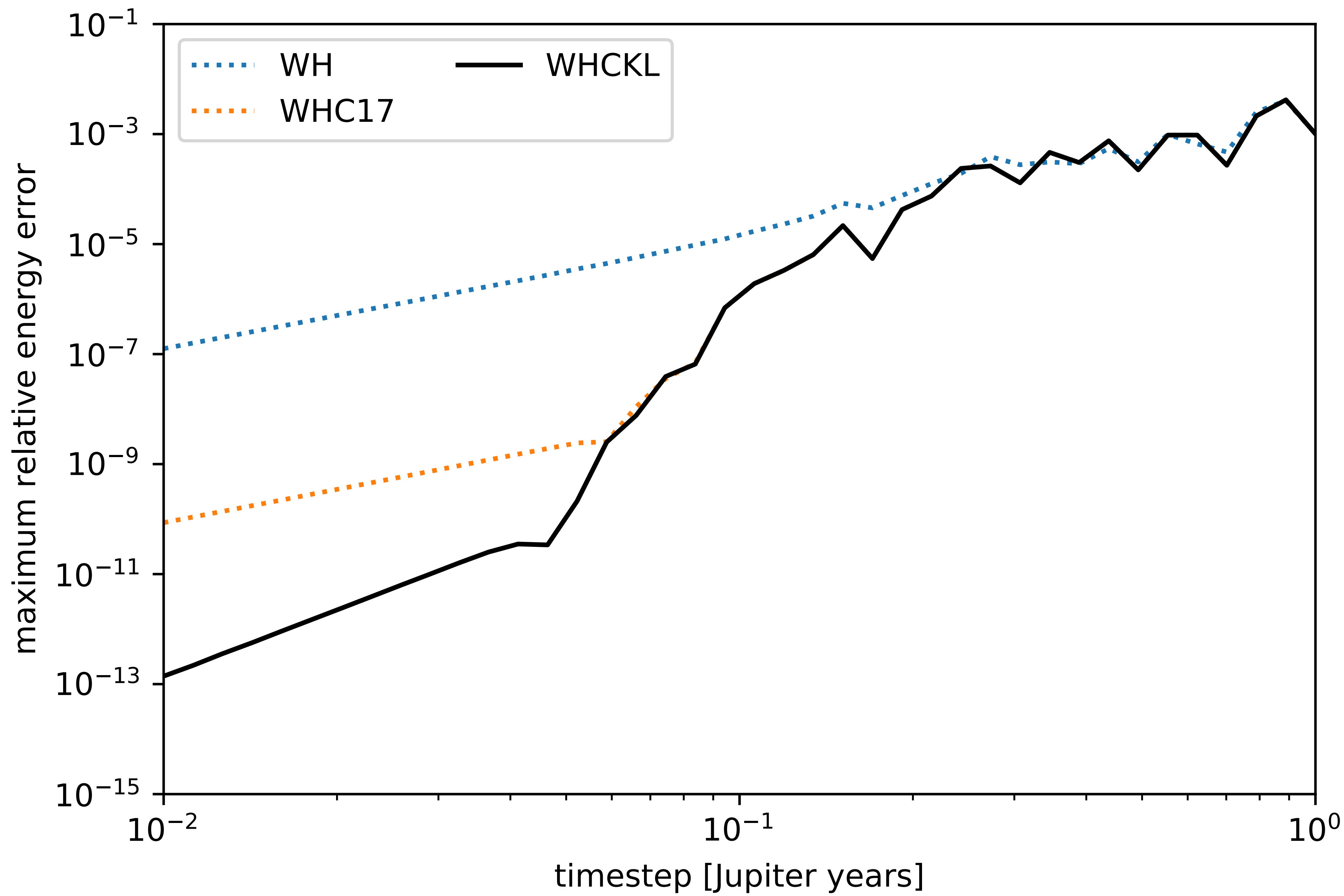
$$H' = A + \epsilon B - \cancel{\epsilon \frac{dt^2}{24}\{A,\{A,B\}\}} - \epsilon^2 \frac{dt^2}{12}\{B,\{A,B\}\} + \mathcal{O}(\cancel{\epsilon dt^4}) + \mathcal{O}(\epsilon^2 dt^4) + \ldots$$



Rein, Tamayo & Brown (2019)

Wisdom-Holman with modified Kernel

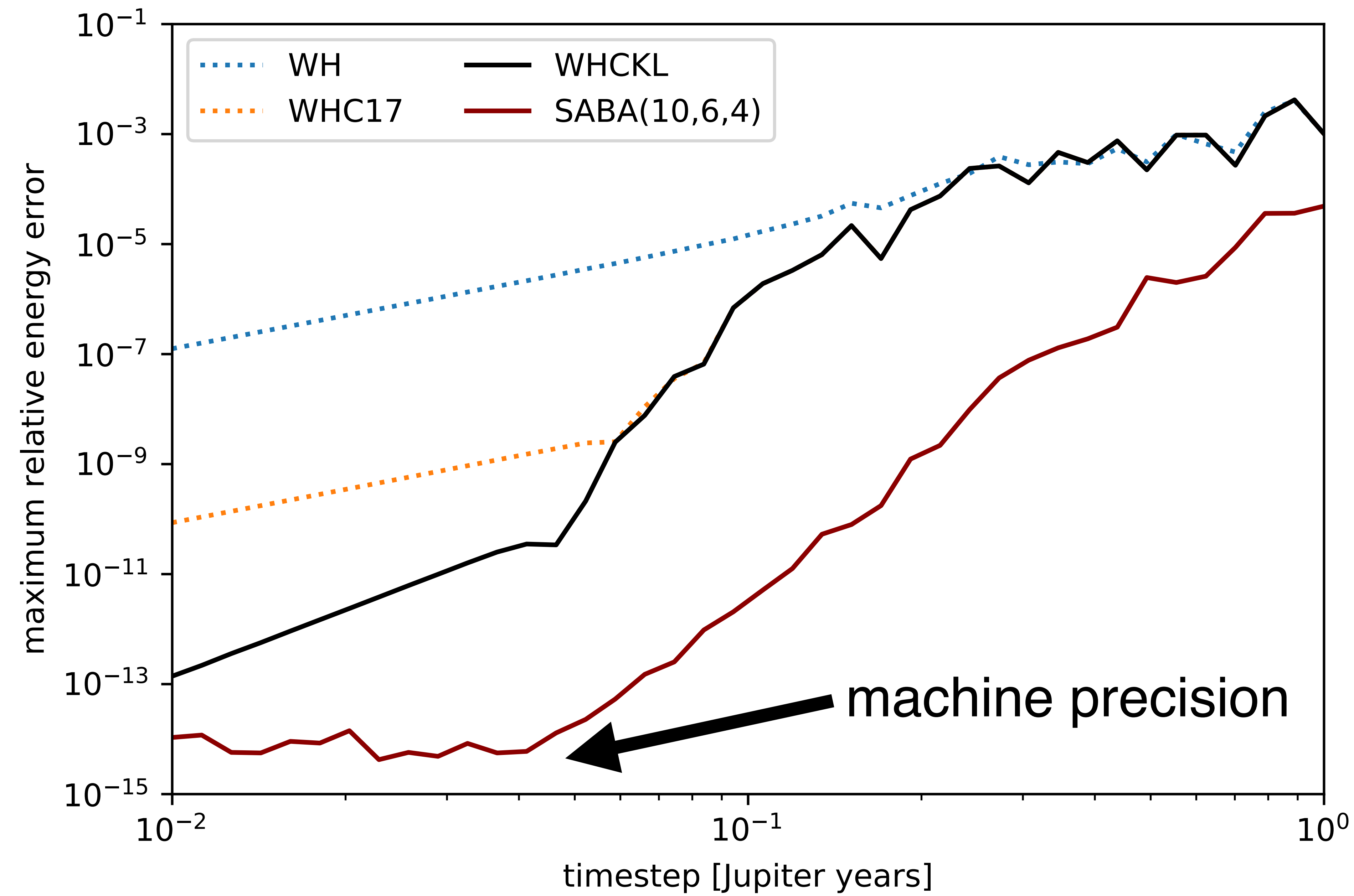
$$H' = A + \epsilon B - \cancel{\epsilon\frac{dt^2}{24}\{A,\{A,B\}\}} - \cancel{\epsilon^2\frac{dt^2}{12}\{B,\{A,B\}\}} + \mathcal{O}(\cancel{\epsilon dt^4}) + \mathcal{O}(\epsilon^2 dt^4) + \ldots$$



Rein, Tamayo & Brown (2019)

SABA(10,6,4)

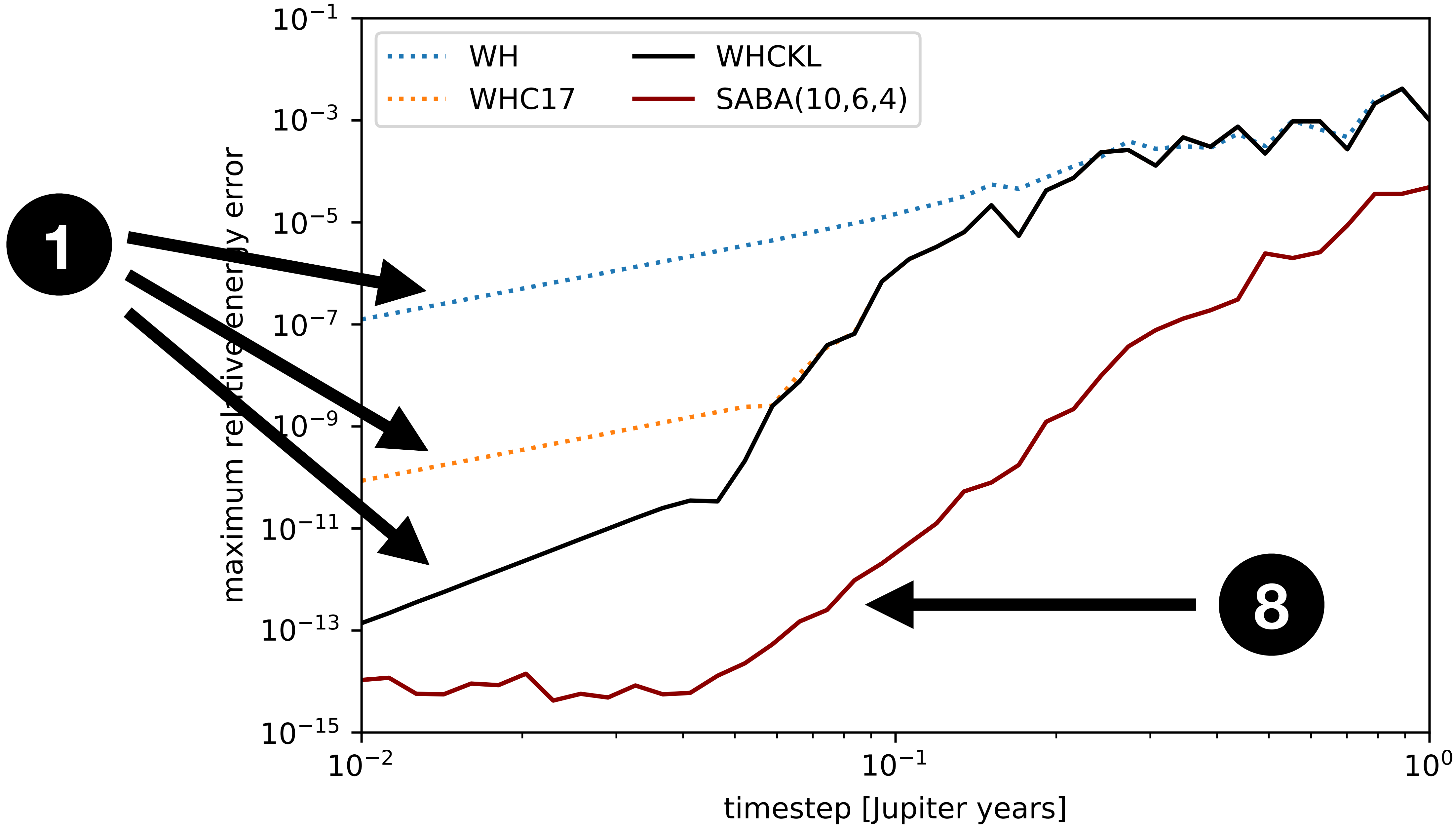
$$H' = A + \epsilon B - \epsilon \cancel{\frac{dt^2}{24}\{A,\{A,B\}\}} - \epsilon^2 \cancel{\frac{dt^2}{12}\{B,\{A,B\}\}} + \mathcal{O}(\cancel{\epsilon dt^4}) + \mathcal{O}(\cancel{\epsilon^2 dt^4}) + \ldots$$



McLachlan (1995), Laskar & Robutel (2001), Blanes et al. (2015)

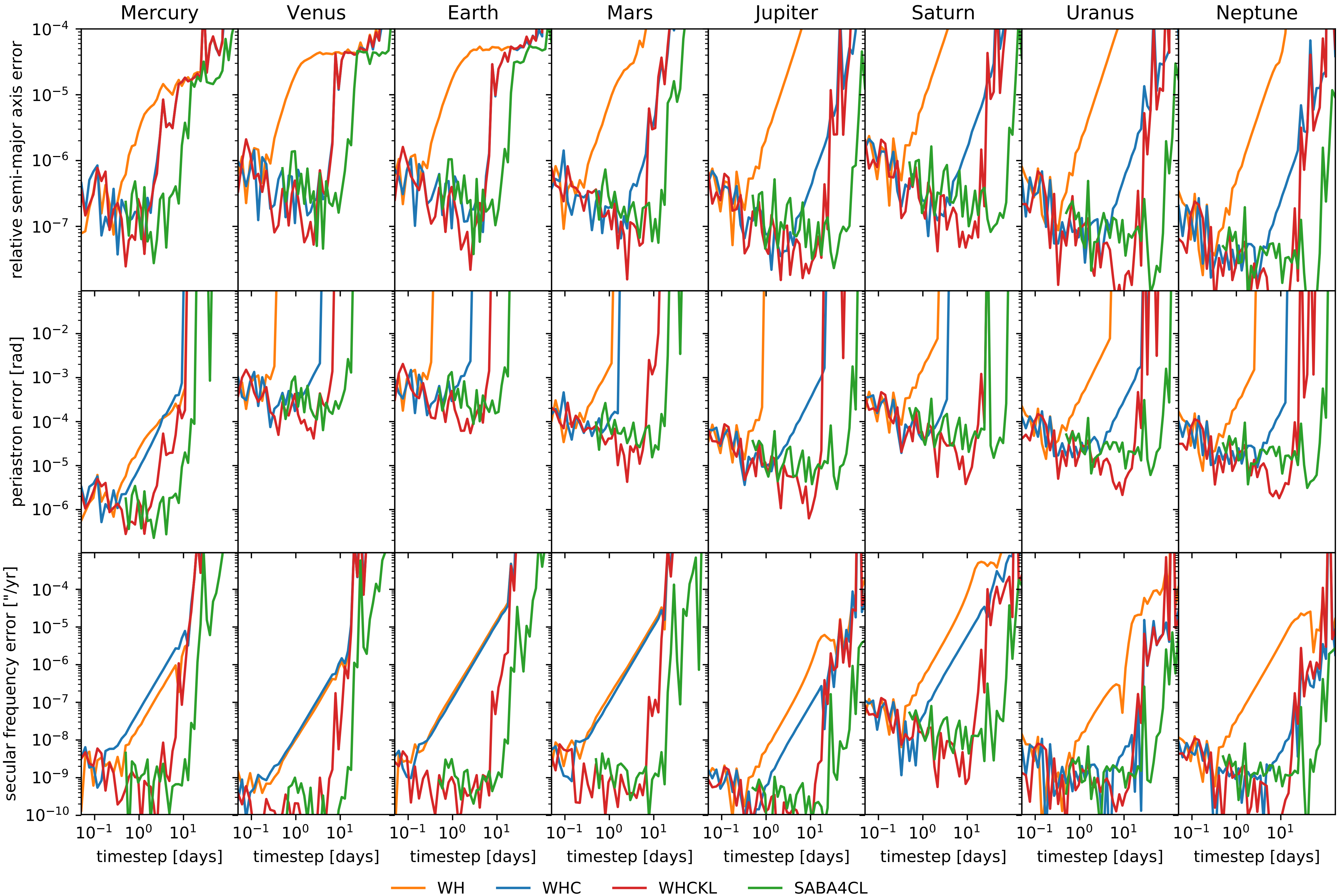
Cost

Measured in function evaluations



Rein, Tamayo & Brown (2019)

Secularly evolving system



Rein, Brown & Tamayo (2019)

Why?

$$
\begin{aligned}
H' =& A + \epsilon B \\
& - \epsilon \frac{dt^2}{24} \{A, \{A, B\}\} \\
& - \epsilon^2 \frac{dt^2}{12} \{B, \{A, B\}\} \\
& + \mathcal{O}(\epsilon dt^4) + \mathcal{O}(\epsilon^2 dt^4) \\
& + \ldots
\end{aligned}
$$

Rein, Brown & Tamayo (2019)

Why?

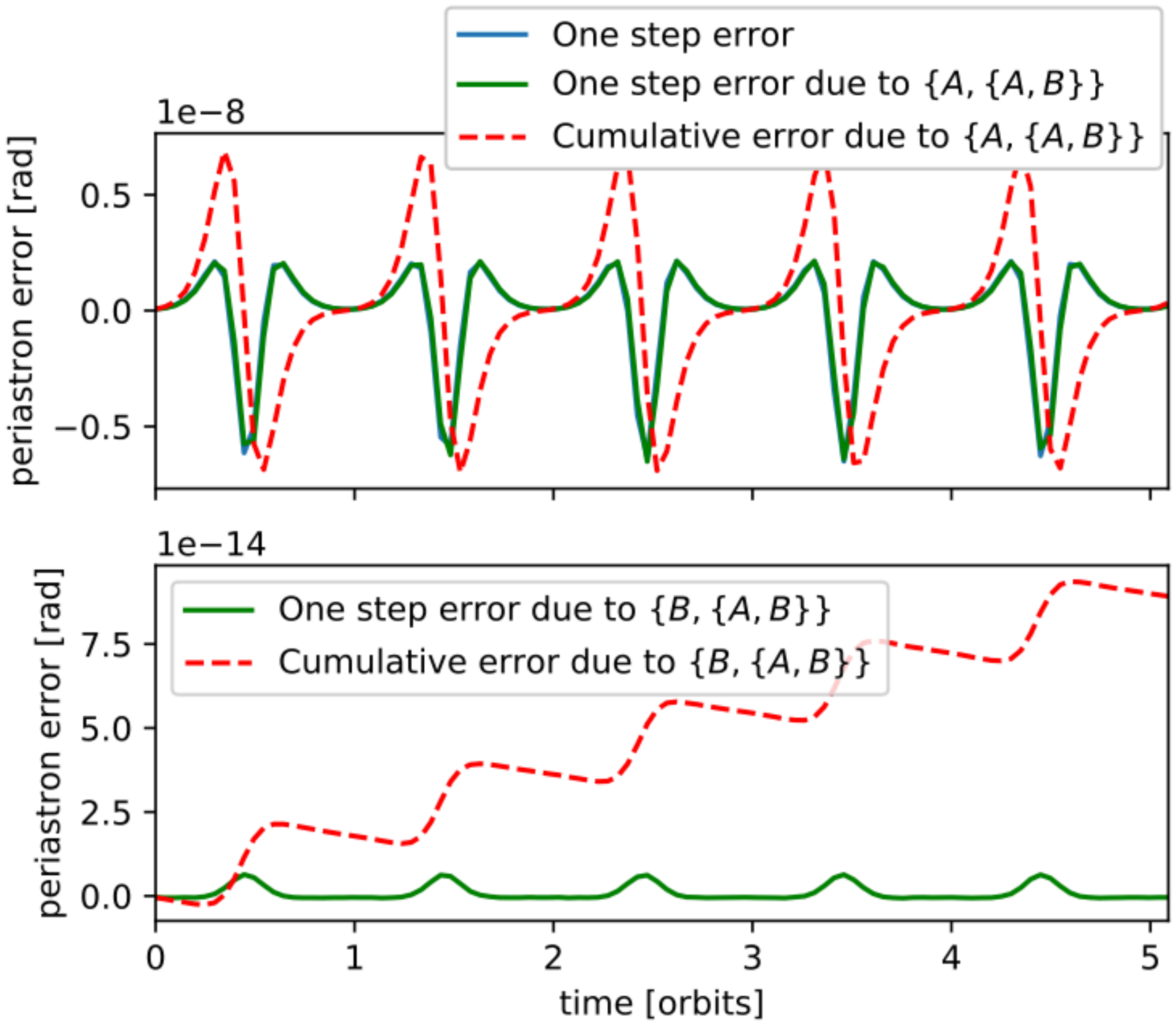


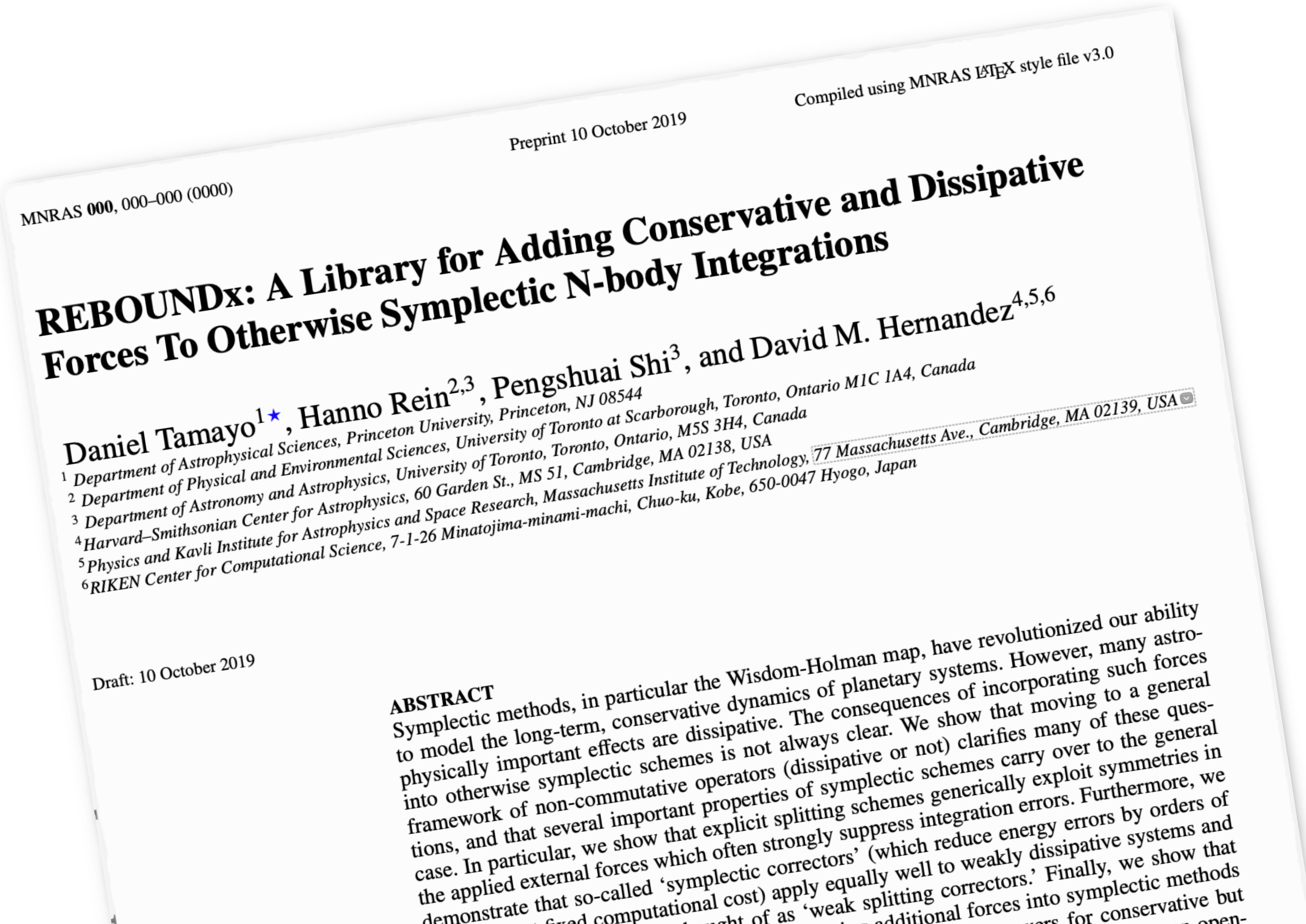
Figure 4. Periastron error in a simulation with one planet and general relativistic corrections. Solid lines are one step errors, dashed lines are cumulative errors.

$$
\begin{aligned}
H' =\, & A + \epsilon B \\
& - \epsilon \frac{dt^2}{24} \{A, \{A, B\}\} \\
& - \epsilon^2 \frac{dt^2}{12} \{B, \{A, B\}\} \\
& + \mathcal{O}(\epsilon dt^4) + \mathcal{O}(\epsilon^2 dt^4) \\
& + \ldots
\end{aligned}
$$

Rein, Brown & Tamayo (2019)

Applications to non-Hamiltonian systems

REBOUNDX



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REBOUNDx: A Library for Adding Conservative and Dissipative Forces To Otherwise Symplectic N-body Integrations

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ABSTRACT
Symplectic methods, in particular the Wisdom-Holman map, have revolutionized our ability to model the long-term, conservative dynamics of planetary systems. However, many astrophysically important effects are dissipative. The consequences of incorporating such forces into otherwise symplectic schemes is not always clear. We show that moving to a general framework of non-commutative operators (dissipative or not) clarifies many of these questions, and that several important properties of symplectic schemes carry over to the general case. In particular, we show that explicit splitting schemes generically exploit symmetries in the applied external forces which often strongly suppress integration errors. Furthermore, we demonstrate that so-called 'symplectic correctors' (which reduce energy errors by orders of

Conclusions

Operator splitting methods are very powerful for integrating perturbed systems

Two methods stand out for planetary systems: WHCKL, SABA(10,6,4)

Energy error is not a good metric for the long-term evolution of secular systems

Symplectic correctors alone offer no improvement to Solar System integrations

Dozens of high order integrators implemented in REBOUND. Orders of magnitude gain in accuracy without an additional cost!